A PRIORI BOUNDS FOR SOME INFINITELY RENORMALIZABLE QUADRATICS: I. BOUNDED PRIMITIVE COMBINATORICS

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Contents

1. Introduction

1.1. **Statement of the main result.** In this paper, we will prove *a priori* bounds for infinitely renormalizable quadratic polynomials of bounded primitive type. Let us begin with recalling some basic definitions of the holomorphic renormalization theory. Most of them can be found in any source on the subject, see [L3, L2, McM1].

A quadratic polynomial $f\colon z \mapsto z^2 + c$ is called *primitively renormalizable with period* p if there exist topological disks $V \Supset U \ni 0$ such that $f^p\colon U \to V$ is a quadratic-like map with connected Julia set, and the domains $f^n U$, $n = 0, 1, \dots, p-1$ are pairwise disjoint. This quadratic-like map is called a *renormalization* of f. If there is an infinite sequence of periods $p_0 < p_1 < \dots$ such that f is primitively p_k-renormalizable then f is called *infinitely primitively renormalizable*. If additionally, there exists a B such that $p_{k+1}/p_k \le B$, $k = 0, 1, \dots$, then f is called infinitely primitively renormalizable of *bounded type*. Such a map has *a priori bounds* if there exists an $\varepsilon > 0$ and a sequence of quadratic-like renormalizations $f^{p_k}\colon U_k \to V_k$ such that $\operatorname{mod}(V_k \smallsetminus U_k) \ge \varepsilon$.

Main Theorem. *Let f be an infinitely primitively renormalizable quadratic polynomial of bounded type. Then f has a priori bounds.*

Stony Brook IMS Preprint #2006/5
August 2006

In the forthcoming notes [KL2], we will prove *a priori* bounds for a class of infinitely renormalizable maps of unbounded type.

For real quadratics of bounded type *a priori bounds* were proved by Sullivan [S], see also [LS, LY, MS]. They were also proved for a class of complex combinatorics of "high bounded type" [L1].

1.2. **Consequences.** *A priori bounds* have numerous consequences. Let us list some of them (below $f_c\colon c\mapsto z^2+c$ stands for an infinitely primitively renormalizable quadratic polynomial of bounded type):

• *The Mandelbrot set is locally connected at c, or equivalently, the polynomial f_c is combinatorially rigid* (see [L1]). The conjecture of local connectivity of the Mandelbrot set (the *MLC Conjecture*) formulated about 20 years ago by Douady and Hubbard (see [DH1]) is a central open problem in holomorphic dynamics. Previously, it was established for all quadratic maps which are not infinitely renormalizable (Yoccoz, see [H]) and for the class of infinitely renormalizable maps of high type mentioned above (see [L1]).

• *The Julia set $J(f_c)$ is locally connected* (see [HJ, J]).

• *The Feigenbaum-Coullet-Tresser Renormalization Conjecture is valid for primitive combinatorics.* This conjecture was established in the work of Sullivan [S], McMullen [McM2] and Lyubich [L2] assuming *a priori* bounds (and thus, unconditionally, for real maps). Now, these results become unconditional for arbitrary primitive combinatorics.

• *Universality and Hairiness of the Mandelbrot set at c.* These properties were conjectured by Milnor [M] and proved in [L2] for maps with *a priori* bounds.

• *The Basic Trichotomy for the measure and Hausdorff dimension of the Julia set $J(f_c)$* which was established in [AL] for maps with *a priori* bounds.

1.3. **Outline of the proof.** We will now give a brief top-down outline of the proof of the Main Theorem.

General strategy: improving of the lengths of the hyperbolic geodesics (§8). Let K_n be the filled Julia set of the renormalization $f^{p_n} : U_n \to V_n$, let

$$\mathcal{K}_n = \bigcup_{i=0}^{p_n-1} f^i(K_n),$$

and let γ_n be the peripheral hyperbolic geodesic in $V \smallsetminus \mathcal{K}_n$ going around K_n. *A priori bounds* are equivalent to the assertion that the hyperbolic length of the γ_n is bounded. Our strategy towards this end is to show that *if the length of some γ_n gets long then it was even longer before*: There exist $M > 0$ and $l > 0$ such that

$$|\gamma_n| > M \Rightarrow |\gamma_{n-l}| > 2M.$$

Pseudo-quadratic-like maps and canonical renormalization (§2). To carry this strategy out, we need a notion of renormalization that respects hyperbolic geometry. To this end we introduce a class of pseudo-quadratic like maps and construct the canonical renormalizations $\mathbf{f}_n : \mathbf{U}_n \to \mathbf{V}_n$ in this class. These renormalizations share the Julia set with the usual quadratic-like renormalizations and have the property that the length of the closed hyperbolic geodesic in the annulus $\mathbf{V}_n \smallsetminus K_n$ is equal to the hyperbolic length of γ_n. Then the above

strategy boils down to showing that for an infinitely renormalizable pseudo-quadratic-like map $\mathbf{f}$ of bounded type, there exist $M > 0$ and $l > 0$ such that

$$|\gamma_l| > M \Rightarrow |\gamma_0| > 2M.$$

Canonical weighted arc diagrams. We then pass from the hyperbolic geometry to the world of extremal length. Using thin-thick decomposition of the Riemann surface $\mathbf{V} \setminus \mathcal{K}_l$, we estimate $|\gamma_l|$ as the sum of extremal widths of thin rectangles crossed by γ_n (§7).

The widths of all thin rectangles that appear in the decomposition can be organized into the *canonical weighted arc diagram*

$$W_{\text{can}} = \sum W(\alpha)\alpha,$$

where α are arcs[1] in $\mathbf{V} \setminus \mathcal{K}_l$ represented by the heights of the thin rectangles and $W(\alpha)$ are their widths. The canonical WAD's possess good functorial properties that are described in §3. The most subtle of these properties is the *domination* relation $\multimap$ that encodes the parallel and series laws for extremal length.

Vertical vs horizontal. The canonical weighted arc diagram can be decomposed into the vertical and horizontal parts, $W^{\text{v}}_{\text{can}}$ and $W^{\text{h}}_{\text{can}}$. The former is composed of the arcs that connect the little Julia sets to the boundary of $\mathbf{V}$; the latter is composed of the arcs connecting the little Julia sets. We notice that if $\|W^{\text{v}}\|_1$ is not too small compared with $\|W^{\text{h}}\|_1$, then the desired improvement of the hyperbolic length follows (§8).

Restrictions of the canonical WAD's and the entropy argument. We consider the nest of the domains $\mathbf{U}^m$, pullbacks of the $\mathbf{V}$ under the iterates of $\mathbf{f}$, and restrict the canonical WAD to these domains. Let X^m stand for the horizontal parts of these restrictions (with appropriate constants subtracted). We show that $f^* X^m \multimap X^{m+1}$. This allows as to conclude that eventually the *diagrams X^n are aligned with the Hubbard tree* (§4) and that

$$\|W^{\text{h}}_{\text{can}} |\, \mathbf{U}^m\| \le \frac{1}{2}\|W^{\text{h}}_{\text{can}}\|, \quad \text{where } m = O(p_l),$$

(§5). The latter property depends on the positivity of entropy of the Hubbard tree dynamics: this is the main place where we use that the renormalizations are of primitive type.

Push-forward via the Covering Lemma (§6). The horizontal width released under restrictions is turned into the vertical width, which implies

$$\|W^{\text{v}}_{\text{can}} |\, \mathbf{U}^m\|_1 \ge \frac{1}{2}\|W^{\text{h}}_{\text{can}} |\, \mathbf{U}^m\|_1, \quad \text{where } m = O(p_l),$$

for the restricted diagrams. To go back to the original diagrams, we push forward the restricted diagram by f^{p_l} using the Covering Lemma of [KL1]. This completes the proof.

1.4. **Terminology and Notation.** We let:
$\mathbb{N} = \{1, 2, \dots\}$ be the set of natural numbers; $\mathbb{Z}_{\ge 0} = \mathbb{N} \cup \{0\}$;
$\mathbb{D} = \{z : |z| < 1\}$ be the unit disk, $\mathbb{T}_r = \{z : |z| = r\}$, $\mathbb{T} = \mathbb{T}_1$;
$\mathbb{A}(r, R) = \{z : r < |z| < R\}$.

A *topological disk* means a simply connected domain in some Riemann surface S. We will say a subset K of $\mathbb{C}$ is an FJ-set ("filled Julia set") if K is compact, connected, and full.

[1]An *arc* is a non-trivial homotopy class of properly embedded paths.

1.5. **Acknowledgment.** This paper was written in collaboration with Mikhail Lyubich following the mathematical work of the author. The primary result for this paper will be combined with the main result of a subsequent paper of Mikhail Lyubich and the author to form one result that is the goal of the series of papers of which this paper is the first.

This work has been partially supported by the Clay Mathematics Institute and the Simons Foundation. Part of it was done during the author's visit to the IMS at Stony Brook and the Fields Institute in Toronto. The author is thankful to these Institutions and Foundations.

2. Pseudo-quadratic-like maps and canonical renormalization

2.1. **Quadratic-like maps and their renormalizations.** For the basic theory of quadratic-like maps, see [DH2].

A *quadratic-like map* is a double branched covering $f : U \to V$, where U and V are topological disc in $\mathbb{C}$, and $U \Subset V \neq \mathbb{C}$. Such a map has a unique critical point, which will be placed at the origin. The *filled Julia set* $K(f)$ is the set of non-escaping points:

$$K(f) = \{z : f^n z \in U,\ n = 0, 1, \dots\}.$$

It is either connected or a Cantor set depending on whether $0 \in K(f)$ or not. (In what follows, we will often skip "filled", since we will never deal with the actual "Julia sets".)

Two quadratic-like maps are called *hybrid equivalent* if they are quasi-conformally conjugate by a map h with $\bar{\partial} h = 0$ a.e. on $K(f)$. The Douady and Hubbard Straightening Theorem asserts that any quadratic-like map with connected Julia set is hybrid equivalent to a quadratic-like restriction of a unique quadratic polynomial.

A quadratic-like map f is called *primitively renormalizable with period* p if there exists a topological disk $U' \ni 0$ such that the domains $f^k(U')$, $k = 0, 1, \dots, p-1$, are contained in U and pairwise disjoint, and $g = f^p : U' \to f^p(U')$ is a quadratic-like map with connected Julia set. In this case, $K_0 = K(g)$, as well as $K_j = f^j(K')$, $j = 0, 1, \dots, p-1$, are called the *little Julia set*, and the invariant set

$$\mathcal{K} = \bigcup_{j=0}^{p-1} K_j$$

is called the *cycle of little Julia sets*.

2.2. **Holomorphic correspondences and d-valued immersions.** In what follows Riemann surfaces are not assumed to be connected.

2.2.1. *Holomorphic correspondences.* Let U and V be two Riemann surfaces. A *holomorphic correspondence* $g\colon U \to V$ is a pair (g_l, g_r) of non-constant holomorphic maps $g_l\colon G \to U$ and $g_r\colon G \to V$, where $G \equiv \mathrm{Graph}_g$ is a Riemann surface.

Non-constant holomorphic maps and their (multivalued) inverse are naturally interpreted as holomorphic correspondences.

Somewhat abusing rigor, we will often identify holomorphic correspondences with the associated "multi-valued maps" ($\equiv$ "polymorphisms") $g = g_r \circ g_l^{-1}$.

2.2.2. *Pullback.* Let $g: G \to V$ and $h: H \to V$ be two non-constant holomorphic maps between Riemann surfaces. Let us consider an analytic set

$$P = \{(x, y) \in G \times H : \ g(x) = h(y)\},$$

and let $p_l: P \to G$ and $p_r: P \to H$ be the projections of P to its components. We call $(p_l, p_r): P \to (G, H)$ the *pre-pullback* of g and h.

Let $S \subset P$ be the set of singular points of P. Let $q: \Pi \to P$ be the desingularization of P, i.e., a Riemann surface Π together with a proper holomorphic map q which is one-to-one on $p^{-1}(P \smallsetminus S)$. Let $\pi_l = p_l \circ q$ and $\pi_r = p_r \circ q$ be the natural projections from Π to G and H respectively. The holomorphic correspondence (π_l, π_r) is called the *pullback* of g and f. It satisfied the following universality property:

Lemma 2.1. *Let F be a Riemann surface, and let $f_l: F \to G$ and $f_r: F \to H$ be two holomorphic maps such that $g \circ f_l = h \circ f_r$. Then there exists a unique holomorphic map $\phi: F \to \Pi$ such that $f_l = \pi_l \circ \phi$ and $f_r = \pi_r \circ \phi$.*

Remark. Note that if one of the maps, g or h, is immersion then the pre-pullback P is non-singular and thus coincides with the pullback. In fact, this will the only case of interest for us.

There is a general principle that nice properties of g (respectively h) are inherited by π_r (respectively, π_l) of the pullback. Here is a statement of this kind which is relevant to our discussion:

Lemma 2.2. *Let $(\pi_l, \pi_r): \Pi \to (G, H)$ be the pullback of $g: G \to V$ and $h: H \to V$.*

- *If g is an immersion then π_r is an immersion.*
- *If h is proper then π_l is proper of the same degree.*

Proof. Let us consider the pre-pullback $(p_l, p_r): P \to (G, H)$ of g and h. Since the map $p: \Pi \to P$ is locally injective, it is sufficient to check that $p_r: P \to H$ is locally injective.

Let $(x_0, y_0) \in P$. Since g is an immersion, there exists a neighborhood $\Omega \subset G$ of x_0 such that $g \,|\, \Omega$ is injective. Take a neighborhood $Q \subset H$ of y_0 such that $h(Q) \subset g(\Omega)$. Then

$$N = \{(x, y) \in \Omega \times Q : \ g(x) = h(y)\} \tag{2.1}$$

is a neighborhood of (x_0, y_0) in P such that the projection $p_r|_N: N \to Q$ is injective (since $x \in \Omega$ is uniquely determined by $y \in Q$).

Let us pass to the second assertion. Since $q: \Pi \to P$ is a proper map of degree one, it is sufficient to check that $p_l: P \to G$ is a proper map of degree $d = \deg h$. (For proper maps between singular curves, the degree is still understood as the number of preimages of a generic point.)

Let $Q \subset G$ be a compact set. Then

$$\begin{aligned} p_l^{-1}(Q) &= \{(x, y) \in Q \times H : \ g(x) = h(y)\} \\ &= \{(x, y) \in Q \times h^{-1}(gQ) : \ g(x) = h(y)\}, \end{aligned}$$

which is a closed subset of a compact set $Q \times h^{-1}(gQ)$. Moreover, for a generic x, the equation $h(y) = h(x)$ has d solutions, so that p_l has degree d. □

Of course, a holomorphic map h is proper if and only if it is a branched cover of finite degree.

2.2.3. *Composition.* Let us consider two holomorphic correspondences, $g = (g_l, g_r)\colon G \to (U, V)$ and $h = (h_l, h_r)\colon H \to (V, W)$. Let us consider the pullback $(\pi_l, \pi_r)\colon \Pi \to (G, H)$ of g_r and h_l.

The *composition* $f = h \circ g$ is defined as the pair $(f_l, f_r)\colon \Pi \to (U, W)$, where $f_l = g_l \circ \pi_l$ and $f_r = h_r \circ \pi_r$. The *inverse* of a correspondence $f = (f_l, f_r)\colon F \to (U, V)$ is just $f^{-1} = (f_r, f_l)\colon F \to (V, U)$. We can also define an identity correspondence $\mathrm{id} = (\mathrm{id}, \mathrm{id})\colon V \to V$ for any Riemann surface V. Thus we can form the category of Riemann surfaces with holomorphic correspondences as morphisms. Unfortunately, it is not an invertible category: the composition $f \circ f^{-1}$ is not always the identity. (The composition $f \circ f^{-1}$ will always contain some component that acts like the identity on a subsurface of U).

2.2.4. *Multi-valued immersions.* Let us consider a holomorphic correspondence $g = (g_l, g_r)\colon G \to (U, V)$. Critical values of g_r are called *critical values* of g, while critical values of g_l are called *ramification points* of g. Accordingly, g is called *unramified* if g_l is an immersion, and g is called *unbranched* or a *multi-valued immersion* if g_r is an immersion.

In the classical language, the associated multi-valued function $\phi = g_r \circ g_l^{-1}$ is unramified if all its local branches are single-valued, and it is a multi-valued immersion if its local branches have the form $\psi(z^{1/n})$, where the ψ are univalent.

A multi-valued immersion g is called *evenly-valued* if g_l is a branched covering. If $\deg g_l = d$, it is called *d-valued*. For a d-valued immersion, the associated multi-valued function ϕ satisfies the following properties:

- local branches of ϕ have the form $\psi(z^{1/n})$, where ψ is univalent;
- ϕ admits an analytic continuation along any path that avoids ramification points;
- outside ramification points, ϕ has d local branches.

Vice versa, if we have a multi-valued function $\phi\colon V \to U$ satisfying these properties, then it lifts to an immersion $i\colon G \to U$, where G is a d-sheeted Riemann surface over V, i.e., there is a branched covering $\pi\colon G \to V$ of degree d. (Points of G are germs of local branches of ϕ.) The holomorphic correspondence $(\pi, i)\colon G \to (V, U)$ is a d-valued immersion.

Lemma 2.3. *If $g\colon G \to (U, V)$ is a d_1-valued immersion and $h\colon H \to (V, W)$ is a d_2-branched immersion then $h \circ g$ is a $d_1 d_2$-valued immersion.*

Proof. Let $(\pi_l, \pi_r)\colon \Pi \to (G, H)$ be the pullback of g_r and h_l. By Lemma 2.2, π_l is a branched covering of degree d_2 and π_r is an immersion. Hence $g_l \circ \pi_l$ a branched covering of degree $d_1 d_2$ and $h_r \circ \pi_r$ is an immersion. $\square$

Remark. It can also be easily checked using the definition of a d-valued immersion in the classical language.

2.2.5. *Invariant sets.* Let $g = (g_l, g_r)\colon G \to (U, V)$ be a holomorphic correspondence, $K \subset U$. We say that K is invariant if $g_l^{-1}(K) = g_r^{-1}(K)$.

2.3. **Pseudo-polynomial-like maps.** Suppose that $\mathbf{U}'$, $\mathbf{U}$ are disks, and $i\colon \mathbf{U}' \to \mathbf{U}$ is a holomorphic immersion, and $f\colon \mathbf{U}' \to \mathbf{U}$ is a degree d holomorphic branched cover. Suppose further that there is an FJ-set $K \subset \mathbf{U}$ that is (i, f)-invariant, and suppose that $K' \equiv i^{-1}(K) = f^{-1}(K)$ is connected. (Note that K' is connected if and only if all the branch values of f lie in K. Also note that, because f is proper, K' is an FJ-set.) Then we say that

$F = (i, f)\colon \mathbf{U}' \to (\mathbf{U}, \mathbf{U})$ is a *ψ-polynomial-like*[2] (*ψ-pl*) map of degree d with filled Julia set K.[3] (We will see that $K \subset \mathbf{U}$ is uniquely determined by F).

We can view F as a holomorphic correspondence $(i, f)\colon \mathbf{U}' \to \mathbf{U}$. Then it follows directly from the definitions that F is a ψ-polynomial-like map (of degree d with filled Julia set K) iff the inverse correspondence F^{-1} is a d-valued holomorphic immersion such that the filled Julia set K is completely invariant under F (and F^{-1} is ramified only on K).

Lemma 2.4. *Let $F = (i, f)\colon \mathbf{U}' \to \mathbf{U}$ be a ψ-polynomial-like map of degree d with filled Julia set K. Then i is an embedding in a neighborhood of $K' \equiv f^{-1}(K)$, and the map $g \equiv f \circ i^{-1}\colon U' \to U$ near K is polynomial-like of degree d.*

Moreover, the domains U and U' can be selected in such a way that $\mathrm{mod}(U \setminus i(U')) \geq \mu(d, \mathrm{mod}(\mathbf{U} \setminus K))$, *where $\mu(d, \nu) > 0$ for $\nu > 0$.*

Proof. Let us show that i is an embedding in a neighborhood of K', and that $i(K') = K$. To this end let us consider the annuli $\mathbf{U} \setminus K$ and $\mathbf{U}' \setminus K'$, and let us uniformize them by the round annuli:

$$\phi\colon \mathbb{A}(1, r) \to \mathbf{U} \setminus K, \quad \phi'\colon \mathbb{A}(1, r') \to \mathbf{U}' \setminus K', \quad \text{where } r' = r^{1/d}.$$

Let $I = \phi^{-1} \circ i \circ \phi'\colon \mathbb{A}(1, r') \to \mathbb{A}(1, r)$. Since $i(K') \subset K$, the map I is proper near the unit circle $\mathbb{T}$. By the Reflection Principle, it admits an analytic extension to a map $\mathbb{A}(1/r', r') \to \mathbb{A}(1/r, r)$ that restricts to a covering $\mathbb{T} \to \mathbb{T}$ of some degree $k > 0$. (We will use the same notation, I, for this extension).

Let us consider the geodesic $\omega' = \mathbb{T}_{\sqrt{r'}}$ in $\mathbb{A}(1, r')$, and let $\omega = I(\omega')$. Since for given $r, r' > 0$, the family of holomorphic immersions $\mathbb{A}(1, r') \to \mathbb{A}(1, r)$ of positive degree is compact, the distance from ω to $\mathbb{T}$ is greater than $\rho = \rho(r, k) > 0$. Let $\Lambda = \mathbb{A}(1/\rho, \rho)$ and let Λ' be the component of $I^{-1}(\Lambda)$ containing $\mathbb{T}$. By the Argument Principle, the map $I\colon \Lambda' \to \Lambda$ is a covering of degree k.

In fact, $k = 1$. Indeed, let γ and γ' be the outer boundaries of Λ and Λ' respectively. Let $\Gamma = \phi(\gamma)$, $\Gamma' = \phi'(\gamma')$, and let Ω and Ω' be the disks bounded by Γ and Γ' respectively. Since $I\colon \gamma' \to \gamma$ is a covering of degree k, so is $i\colon \Gamma' \to \Gamma$. Hence $i\colon \Omega' \to \Omega$ is a branched covering of degree k. If $k > 1$, i would necessarily have a critical point, contradicting the assumption that it is an immersion.

Furthermore, let $F = \phi^{-1} \circ f \circ \phi'\colon \mathbb{A}(1, r') \to \mathbb{A}(1, r)$. Since it is a covering map of degree d, $F(z) = z^d$. Hence $F^{-1}(\mathbb{A}(1, \rho)) = \mathbb{A}(1, \rho')$, where $\rho' = \rho^{1/d}$. Notice that the hyperbolic distance from $\mathbb{T}_\rho$ to $\mathbb{T}$ in $\mathbb{A}(1/r, r)$ is equal to the hyperbolic distance from $\mathbb{T}_{\rho'}$ to $\mathbb{T}$ in $\mathbb{A}(1/r', r')$. Since the map $I\colon \mathbb{A}(1/r', r') \to \mathbb{A}(1/r, r)$ contracts the respective hyperbolic metrics by a uniform factor, we have: $I(\mathbb{T}_{\rho'}) \subset \mathbb{A}(1, \lambda\rho)$, where $\lambda = \lambda(r, d, \rho) < 1$. (Actually $\lambda = \lambda(r, d)$ because $\rho = \rho(r, d)$.) Putting all these together, we conclude that

$$W' \equiv I \circ F^{-1}(\mathbb{A}(1, \rho)) \subset \mathbb{A}(1, \lambda\rho) \equiv W.$$

and moreover, the map $F \circ I^{-1}\colon W' \to W$ is a covering of degree d.

Letting

$$U' = K \cup \phi(W'), \quad U = K \cup \phi(W),$$

[2] Pronounced "pseudo-polynomial-like"

[3] Strictly speaking, we have defined a "ψ-polynomial-like map with connected Julia set".

we see that the map $f \circ i^{-1}\colon U' \to U$ is polynomial-like of degree d and $\operatorname{mod}(U \smallsetminus U') \geq \dfrac{1}{2\pi} \log \lambda$, where λ depends only on d and $\operatorname{mod}(\mathbf{U} \smallsetminus K)$. $\square$

(The above estimates with poincare metric also show that if $\hat{K} \subset \mathbf{U}$ is compact, and $K \subset \hat{K}$, and $(\hat{K}_i)_{i=0}^{\infty}$ is defined by $\hat{K}_0 = \hat{K}$ and $\hat{K}_{i+1} = i(f^{-1}(\hat{K}))$, then $K = \bigcap_{i=0}^{\infty} \hat{K}_i$. Therefore K is uniquely determined by F).

Lemma 2.3 implies that the n-fold iterate of a ψ-PL map of degree d is a ψ-PL map of degree d^n. We denote it

$$F^n = (i_n, f^n)\colon \mathbf{U}^n \to \mathbf{U}.$$

The Julia set $K = K(f)$ is embedded into all the domains $\mathbf{U}^n$, and we will keep the same notation for the embedded sets.

Notice also that there is a natural ψ-PL map $\mathbf{U}^n \to \mathbf{U}^{n-1}$, the "top" of the pullback triangle corresponding to the n-fold composition of F. We call it the *restriction* of F to $\mathbf{U}^n$ and use for it the same notation $F = (i, f)$.

We will also use a loose notation $f : \mathbf{U} \to \mathbf{U}$ for the ψ-ql map $(f, i) : \mathbf{U}' \to \mathbf{U}$, identifying $f \circ i^{-1}$ with f and viewing it as a multivalued map on $\mathbf{U}$.

2.4. **Canonical Renormalization.** A ψ-polynomial-like map of degree 2 is called ψ-*quadratic-like* (ψ*-ql*) . More generally, a ψ-PL map $F = (i, f)$ of degree d is called *unicritical* if it has a single critical point (so that f has local degree d at this point). We will abbreviate unicritical ψ-PL maps as $u\psi$-PL maps.

A $u\psi$-PL map $F = (i, f)$ with connected Julia set is called *renormalizable* if its polynomial-like restriction to a neighborhood of $K(F)$ is renormalizable. If F is primitively renormalizable, then one can define a *canonical* renormalization $G = R(F) = (j, g)$ in the space of $u\psi$-PL maps as follows.

Let us first consider a polynomial-like renormalization $g\colon U' \to V'$ of f near the Julia set. Let $\mathcal{K} = \mathcal{K}(g) = \cup_{i=0}^{p-1} K(g)$, where p is the renormalization period. Because g is a primitive renormalization, we can assume that $V' \cap \mathcal{K}(g) = K(g)$. Let us consider a Jordan loop $\gamma = \partial U'$ and the associated covering $\pi\colon \Omega \to V \smallsetminus \mathcal{K}(g)$.

Lemma 2.5. *Ω is homeomorphic to the space of paths $\delta\colon [0, 1] \to V \smallsetminus \mathcal{K}(g)$ such that $\delta(0) \in V'$, modulo homotopy fixing $\gamma(1)$.*

Proof. Because $V' \smallsetminus K(g)$ deformation retracts to γ, the above space of paths modulo the homotopy is identified with the space $\mathcal{P}$ of paths $\delta\colon [0, 1] \to V \smallsetminus \mathcal{K}(g)$ such that $\delta(0) \in \gamma$, modulo homotopy fixing $\delta(1)$. Let $\sim$ denote this homotopy equivalence on $\mathcal{P}$.

Since π is associated with γ, the loop γ lifts to a loop $\hat{\gamma} \subset \Omega$ that projects homeomorphically onto γ.

Any path $\delta \in \mathcal{P}$ can be lifted to a path $\hat{\delta}$ on Ω such that $\hat{\delta}(0) \in \hat{\gamma}$. Furthermore, any homotopy $\delta_t \in \mathcal{P}$ lifts to a homotopy $\hat{\delta}_t$ satisfying the boundary conditions: $\hat{\delta}_t(0) \in \hat{\gamma}$ and $\hat{a} \equiv \hat{\delta}_t(1)$ is fixed. Hence $\hat{a}$ is a well defined point of $\hat{V}$ associated to the homotopy class $[\delta] \in \mathcal{P}/\sim$.

Vice versa, given a point $\hat{a} \in \Omega$, all paths $\hat{\delta}$ joining it to $\hat{\gamma}$ are homotopic (preserving the boundary conditions) and hence push down to a homotopy class $[\delta] \in \mathcal{P}/\sim$ of paths joining $a = \pi(\hat{a})$ to γ. This determines the inverse map $\Omega \to \mathcal{P}/\sim$. $\square$

Let us consider a topological annulus $A = \bar{V}' \smallsetminus K(g)$. It lifts to a topological annulus $\hat{A} \subset \Omega$ that deformation retracts to $\hat{\gamma}$ and $\pi\colon \hat{A} \to A$ is an isomorphism. Hence

$$\Omega \approx \Omega \sqcup_\pi A.$$

The latter Riemann surface naturally embeds into

$$\hat{V} = \Omega \sqcup_\pi V',$$

and the covering π naturally extends to a projection $\hat{V} \to V$ (which still will be denoted by π). In what follows we will identify V' (in particular, $K(g)$) with its homeomorphic lift to $\hat{V}$.

Lemma 2.6. *The map g^{-1} lifts to a d-valued immersion $\hat{g}^{-1}$ on $\hat{V}$ (ramified only at $g(0)$). The inverse correspondence is a $u\psi$-PL map with the only critical point at 0.*

Proof. Take a path $\delta\colon [0,1] \to V \smallsetminus \mathcal{K}(g)$ such that $\delta(0) \in V'$. Then $g^{-1}(\delta(0))$ consists of d points $b_i \in U'$. By Lemma 2.3 F^{-p} is a multi-valued immersion on V, so that it satisfies the path-lifting property. Hence δ lifts by F^{-p} to d path γ_i originating at the points b_i.

Let us now consider a homotopy δ_t fixing $\delta(1)$ and moving $\delta(0)$ within $V' \smallsetminus K(g)$. Since $V \smallsetminus \mathcal{K}(g)$ does not contain the critical values of F^p, it lifts to a homotopy of each path γ_i fixing $\gamma_i(1)$ and moving $\gamma_i(0)$ within $U' \smallsetminus K(g)$. By Lemma 2.5, it represents a well-defined point $\hat{a}_i \in \Omega$.

This defines a d-valued lift $\hat{g}^{-1}$ of $F^{-p}\colon V \smallsetminus \mathcal{K}(g) \to V \smallsetminus \mathcal{K}(g)$ to Ω. Since F^{-p} is unramified immersion on $V \smallsetminus \mathcal{K}(g)$ satisfying the path-lifting property, so is $\hat{g}^{-1}$.

Since g^{-1} is a d-valued immersion on $K(g)$ (ramified only at $g(0)$), $\hat{g}^{-1}$ extends as d-valued immersion through $K(g)$ (ramified only at $g(0)$).

Since near $K(g)$, $\hat{g}^{-1}$ is the inverse to a quadratic-like map f, the conclusion follows. $\square$

3. Canonical weighted arc diagram

3.1. **Arc diagrams.** Let S be a hyperbolic open Riemann surface of finite topology without cusps. It is conformally equivalent to the interior of a compact Riemann surface $\mathbf{S}$ with non-empty boundary.[4] The boundary of $\mathbf{S}$ is called the *ideal boundary* of S. It is canonically attached to S in the sense that any conformal isomorphism $S \to S'$ extends to a homeomorphism $\mathbf{S} \to \mathbf{S}'$.

A *path* in some topological space Z is an embedded interval $\gamma\colon I \to Z$. It is called *open, closed, or semiclosed* depending on the nature of I. An open path $\gamma\colon (0,1) \to S$ is called *proper* if it extends to a closed path $\boldsymbol{\gamma}\colon [0,1] \to \mathbf{S}$ such that $\boldsymbol{\gamma}\{0,1\} \subset \partial\mathbf{S}$. Two proper paths γ_0 and γ_1 in S are called *properly homotopic* if there is a homotopy $\boldsymbol{\gamma}_t$, $t \in [0,1]$, connecting $\boldsymbol{\gamma}_0$ to $\boldsymbol{\gamma}_1$ through a family of proper paths.[5]

An *arc* on S is a class of properly homotopic paths, $\alpha = [\gamma]$. An arc is called *trivial* if it has representing paths $\boldsymbol{\gamma}\colon I \to \mathbf{S}$ in arbitrary small neighborhoods of $\partial\mathbf{S}$. Let $\mathcal{A}(S)$ stand for the set of all non-trivial arcs on S.

Two different arcs are said to be *non-crossing* if they can be represented by non-crossing paths. An *arc diagram* is a family of pairwise non-crossing arcs α_i. Note that any arc diagram consists of at most $3|\chi(S)|$ arcs, where $\chi(S)$ is the Euler characteristic of S.

[4]In what follows, all Riemann surfaces are assumed to be of this type, unless otherwise is explicitly stated.
[5]Note that this homotopy is automatically isotopy.

A *weighted arc diagram (WAD)* on S is an arc diagram $\mathbf{a} = \{\alpha_i\}$ endowed with weights $w_i \in \mathbb{R}_+$. In this case, the arc diagram $\mathbf{a}$ is called the *support* of W.[6] Let $\mathcal{W}(S)$ stand for the set of WAD's on S.

The set $\mathcal{W}(S)$ is *partially ordered*: $X \le Y$ is $X(\alpha) \le Y(\alpha)$ for any $\alpha \in \operatorname{supp} X$. We will also write $X \le Y + c$ if $X(\alpha) \le Y(\alpha) + c$ for any $\alpha \in \operatorname{supp} X$.

The sum of two WAD's, $X + Y$, is well defined whenever any two arcs $\alpha \in \operatorname{supp} X$ and $\beta \in \operatorname{supp} Y$ are either the same or non-crossing. The difference $X - Y$ is always well defined if we let $(X - Y)(\alpha) = 0$ whenever $X(\alpha) \le Y(\alpha)$. Similarly, $X - c$ is well defined for any constant $c \ge 0$.

We will make use of two norms on the space of WAD's:

$$\|W\|_\infty = \sup_{\alpha \in \mathcal{A}} W(\alpha); \quad \|W\|_1 = \sum_{\alpha \in \mathcal{A}} W(\alpha).$$

If $f\colon U \to V$ is a holomorphic covering between two Riemann surfaces then there is a natural *pull-back* operation $f^*\colon \mathcal{W}(V) \to \mathcal{W}(U)$ acting on the WAD's.

A *proper lamination* $\mathcal{F}$ on S is a Borel set $Z \subset S$ explicitly realized as a union of disjoint proper paths called the *leaves* of $\mathcal{F}$. Any proper lamination[7] can be written $\mathcal{F} = \bigcup_\alpha \mathcal{F}(\alpha)$, where $\mathcal{F}(\alpha)$ comprises the leaves of $\mathcal{F}$ that represent α. The arcs $\alpha \in \mathcal{A}$ for which $\mathcal{F}(\alpha)$ is non-empty assemble an arc diagram. Let us weight each of these arcs with the weight $W_\mathcal{F}(\alpha)$ equal to the *extremal width* $\mathcal{W}(\mathcal{F}(\alpha))$ of the sublamination $\mathcal{F}(\alpha)$ (viewed as a path family). In this way we obtain the WAD $W_\mathcal{F} = \sum_\alpha W_\mathcal{F}(\alpha) \cdot \alpha$ corresponding to $\mathcal{F}$.

Note that if $f\colon U \to V$ is a holomorphic covering between two Riemann surfaces and $\mathcal{F}$ is a proper lamination on V then $f^*(\mathcal{F})$ is a proper lamination on U and $W_{f^*(\mathcal{F})} = f^*(W_\mathcal{F})$.

Weighted arc diagrams that are $W_\mathcal{F}$ for some proper lamination $\mathcal{F}$ are called *valid*.

3.2. **Canonical WAD.** Let us consider the universal covering $\pi\colon \mathbb{D} \to \operatorname{int} \mathbf{S}$. Let Γ be the Fuchsian group of deck transformations of π, and let $\Lambda \subset \mathbb{T}$ be its limit set. Since $\mathbf{S}$ has non-empty boundary, Λ is a Cantor set. Moreover, π extends continuously onto $\hat{S} = \bar{\mathbb{D}} \smallsetminus \Lambda$, and the restriction of π to any component I of $\partial \hat{S}$ is a universal covering onto some component I of $\partial \mathbf{S}$.

Let us pick two components, $I \ne J$, of $\partial \hat{S}$. The disk $\mathbb{D}$ with these two intervals as horizontal sides determines a quadrilateral $Q(I, J)$. This quadrilateral can be conformally uniformized, $\phi\colon Q(I, J) \to \mathbf{Q}(a)$, by a standard quadrilateral $\mathbf{Q}(a) = [0, a] \times [0, 1]$ in such a way that I and J correspond to the horizontal sides of $\mathbf{Q}(a)$. The *vertical foliation* $\mathcal{F}(I, J)$ on $Q(I, J)$ is the ϕ-pullback of the standard vertical foliation on $\mathbf{Q}(a)$.

Assume now that $a > 2$, and let us cut off from $\mathbf{Q}(a)$ two side squares, $[0, 1] \times [0, 1]$ and $[a-1, a] \times [0, 1]$. We call the left-over rectangle $\mathbf{Q}_{\mathrm{can}}(a)$, and we let $Q_{\mathrm{can}}(I, J) = \phi^{-1}(\mathbf{Q}_{\mathrm{can}}(a))$. The side quadrilaterals that we have cut off from $Q(I, J)$ are called its *buffers*.

Let $\mathcal{F}_{\mathrm{can}}(I, J)$ be the restriction of $\mathcal{F}(I, J)$ to $Q_{\mathrm{can}}(I, J)$. Obviously, for any deck transformation $\gamma \in \Gamma$, we have:

$$\mathcal{F}_{\mathrm{can}}(\gamma(I), \gamma(J)) = \gamma(\mathcal{F}_{\mathrm{can}}(I, J)). \tag{3.1}$$

Lemma 3.1. *The rectangles $Q_{\mathrm{can}}(I, J)$ are pairwise disjoint.*

[6] We can also think of a WAD as a function $\mathcal{A}(S) \to \mathbb{R}_+$ supported on some arc diagram.

[7] In what follows, all laminations under consideration are assumed to be proper.

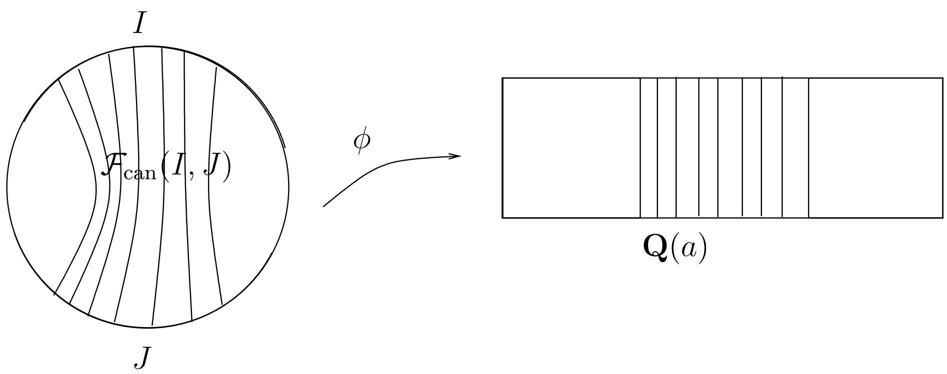

FIGURE 3.1. The canonical foliation

Proof. Let us consider two rectangles, $Q \equiv Q(I, J)$ and $Q' \equiv Q(I', J')$. Then we can find one interval from each pair, say J and J', such that $J \neq J'$ (so $J \cap J' = \emptyset$), and such that there is a component T of $\mathbb{T} - (J \cup J')$ such that $T \cap (I \cup I') = \emptyset$.

We let B be the buffer of $Q(I, J)$ that has a horizontal side (which is a subset of J) that shares an endpoint with T; we define B' likewise. Then if any vertical leaf γ of $\mathcal{F}_{\rm can}(I, J)$ crossed any vertical leaf γ' of $\mathcal{F}_{\rm can}(I', J')$, then every vertical leaf of B would cross every vertical leaf of B'. This would contradict Lemma 10.6. $\square$

This lemma allows us to define the *canonical lamination* $\mathcal{F}_{\rm can}(\hat{S})$ as the union of the laminations $\mathcal{F}_{\rm can}(I, J)$ for all pairs of different components I and J of $\partial\hat{S}$. By (3.1), this lamination is Γ-invariant, and hence it can be pushed forward to $\mathbf{S} \supset S$. In this way we obtain the *canonical lamination on* S:

$$\mathcal{F}_{\rm can}(S) = \mathcal{F}_{\rm can}(\mathbf{S}) = \pi_*(\mathcal{F}_{\rm can}(\hat{S})).$$

The corresponding weighted arc diagram $\alpha \mapsto W_{\rm can}(S, \alpha)$, $\alpha \in \mathcal{A}(S)$, is called the *canonical WAD* on S.[8] By definition, it is valid.

We will now list several basic properties of the canonical WAD. The rest of the theory will be based on these properties in an essentially axiomatic way.

3.3. **Property A: Maximality.** Let $W_{\max}(S)\colon \mathcal{A} \to \mathbb{R}_+$ stand for the functional assigning to an arc $\alpha \in \mathcal{A}$ the extremal width of the family of all proper paths γ in S representing α. (Note that $W_{\max}(S)$ is *not* a WAD as it is not supported on an arc diagram.)

Lemma 3.2. *For any valid arc diagram W on S, we have:*

$$W \leq W_{\max}(S) \leq W_{\rm can}(S) + 2.$$

Proof. The first inequality is obvious, so let us focus on the second one.

It is trivial for any arc $\alpha \in \mathcal{A}$ with $W_{\max}(\alpha) \leq 2$. Let us consider some arc $\alpha \in \mathcal{A}$ with $W_{\max}(\alpha) > 2$. This arc connects two boundary components, σ and ω, of S.

The path family $\mathcal{G}(\alpha)$ representing α lifts to a path family $\hat{\mathcal{G}}(\alpha)$ consisting of all the paths in $\mathbb{D}$ that connect two appropriate arcs on $\mathbb{T}$, I and J, covering σ and ω respectively. Viewing

[8]We will use abbreviated notations $W_{\rm can}(S)$ or $W_{\rm can}(\alpha)$ whenever it does not lead to confusion.

I and J as the horizontal sides of the rectangle $Q(I,J)$ based on $\bar{\mathbb{D}}$, we obtain the desired estimate:

$$\mathcal{W}(\mathcal{G}(\alpha)) \le \mathcal{W}(\hat{\mathcal{G}}(\alpha)) \le \mathcal{W}(Q(I,J)) = W_{\mathrm{can}}(S,\alpha) + 2$$

(where we have made use of Lemma 10.1 for the first estimate). □

3.4. **Property B: Natural behavior under coverings.**

Lemma 3.3. *If $f\colon U \to V$ is a finite-degree covering then $W_{\mathrm{can}}(U) = f^* W_{\mathrm{can}}(V)$.*

Proof. Let $\pi_U\colon \hat{U} \equiv \bar{\mathbb{D}} \smallsetminus \Lambda_U \to U$ and $\pi_V\colon \hat{V} \equiv \bar{\mathbb{D}} \smallsetminus \Lambda_V \to V$ be the universal coverings of U and V, with deck transformations groups Γ_U and Γ_V respectively. Since f has a finite degree, the group Γ_U has a finite index in Γ_V. It follows that $\Lambda_U = \Lambda_V$, so that $\hat{U} = \hat{V}$. Hence $\mathcal{F}_{\mathrm{can}}(\hat{U}) = \mathcal{F}_{\mathrm{can}}(\hat{V}) \equiv \mathcal{F}$. Then

$$\mathcal{F}_{\mathrm{can}}(U) = \mathcal{F}/\Gamma_U = f^*(\mathcal{F}/\Gamma_V) = f^*(\mathcal{F}_{\mathrm{can}}(V)),$$

and the conclusion follows. □

3.5. **Property C: Behavior under partially proper maps.** Let $\mathcal{E}(S)$ stand for the set of ends of the Riemann surface S (that, under our standing assumption, can be identified with the set of boundary components of $\mathbf{S}$). We say that S is *partially marked* if we have chosen a subset $\mathcal{E}_p(S) \subset \mathcal{E}(S)$ of ends that we call "proper".

A map $e\colon U \to V$ between partially marked Riemann surfaces is called *partially proper* if it is proper on proper ends. An arc $\alpha \in \mathcal{A}(S)$ is called *horizontal* if it connects proper ends of S. Let $\mathcal{A}^{\mathrm{h}}(S)$ stand for the set of horizontal arcs on S, and let $\mathcal{W}^{\mathrm{h}}(S)$ be the set of horizontal WAD's on S. The *horizontal canonical WAD* $W^{\mathrm{h}}_{\mathrm{can}}(S)$ is the restriction of the canonical WAD to the set of horizontal arcs.

Notice that a partially proper map $e\colon U \to V$ induces a push-forward map on the horizontal arcs, $e_*\colon \mathcal{A}^{\mathrm{h}}(U) \to \mathcal{A}^{\mathrm{h}}(V)$, and hence, a pullback map on the horizontal arc diagrams, $e^*\colon \mathcal{W}^{\mathrm{h}}(V) \to \mathcal{W}^{\mathrm{h}}(U)$, defined by $e^*(Y)(\alpha) = Y(e_*\alpha)$.

Lemma 3.4. *Let U and V be partially marked Riemann surfaces, and let $e\colon U \to V$ be a partially proper holomorphic map. Then*

$$W^{\mathrm{h}}_{\mathrm{can}}(U) \le e^* W^{\mathrm{h}}_{\mathrm{can}}(V).$$

Proof. Let us consider a horizontal arc $\alpha \in \operatorname{supp} W^{\mathrm{h}}_{\mathrm{can}}(U)$. It connects two proper ends, σ and ω, and it lifts to an arc in $\hat{U}$ connecting some components I and J of $\partial\hat{U}$. Let Q be the quadrilateral based on $\hat{U}$ with the horizontal sides I and J. Let $I' = \hat{e}(I)$, $J' = \hat{e}(J)$, and let Q' be the corresponding quadrilateral based on $\hat{V}$. Then $\hat{e}\colon Q \to Q'$ maps the horizontal sides of Q to the corresponding horizontal sides of Q'. By Corollary 10.3, $\mathcal{W}(Q) \le \mathcal{W}(Q')$. But $\mathcal{W}(Q) = W_{\mathrm{can}}(U,\alpha) + 2$, $\mathcal{W}(Q') = W_{\mathrm{can}}(V, e_*(\alpha)) + 2$, and the desired conclusion follows. □

Similarly, an arc $\alpha \in \mathcal{A}(S)$ is called *vertical* if it connects a proper end of S to an improper one. The *vertical canonical WAD* $W^{\mathrm{v}}_{\mathrm{can}}(S)$ is the restriction of the canonical WAD to the set of vertical arcs.

3.6. **Property D: Domination.** Let us now introduce an important relation between WAD's.

An *integer WAD* $\mathbf{a}$ (IWAD) is a WAD with integer coefficients, that is, a formal linear combination $\sum n_i \alpha_i$, where α_i is a arc diagram, and $n_i \in \mathbb{N}$. It will be also convenient to write $\mathbf{a}$ as a formal sum of arcs, $\mathbf{a} = \sum \alpha_j$, where any two different arcs α_j are non-crossing. The order on WAD's induces a natural order on IWAD's.

Let us consider two Riemann surfaces, $U \subset V$. Given a path γ on V, the restriction $\gamma \cap U$ has only finitely many non-trivial components, $(\gamma_i)_{i=1}^n$. They represent a sequence of arcs on U, $I(\gamma) \equiv (\alpha_i \equiv [\gamma_i])_{i=1}^n$, called the *itinerary* of γ.

We say that a sequence of arcs (α_i) on U *arrows* an arc β on V if there exists a path γ representing β such that $I(\gamma) = (\alpha_i)$. We will use notation $(\alpha_i) \longrightarrow \beta$ for the arrow relation.

Remark. Note that the endpoint of γ_i is connected to the beginning of γ_{i+1} by a path that goes through some component K of $V \smallsetminus U$. In this case, the end of U corresponding to this component is *not properly embedded* into V. This remark is useful as it reduces a number of possibilities of how the arc β can be composed by arcs α_j.

We say that a *IWAD* $\mathbf{a} = \sum \alpha_i$ *arrows* β if the arcs α_i can be ordered so that the string of arcs (α_i) arrows β. (In other words, we "abelianize" the arrow relation.) We use the same notation, $\mathbf{a} \longrightarrow \beta$, for this arrow relation.

Let us now consider two WAD's, $X \in \mathcal{W}(U)$ and $Y \in \mathcal{W}(V)$. We say that X *dominates* Y, written

$$X \multimap Y,$$

if we can write

$$X \geq \sum_i \sum_j w_{ij} \alpha_{ij}, \quad Y = \sum_i v_i \beta_i$$

where, for each i,

$$(\alpha_{ij})j \longrightarrow \beta_i$$

and

$$\bigoplus_j w_{ij} \geq v_i.$$

The basic example comes from laminations on V:

Lemma 3.5. *Given a valid WAD Y on V, there exists a valid WAD X on U such that $X \multimap Y$.*

Proof. Since Y is valid, $Y = W_{\mathcal{F}}$ for some lamination $\mathcal{F}$ on V. For $\beta \in \operatorname{supp} W_{\mathcal{F}}$, let $\mathcal{F}(\beta)$ be the sublamination of $\mathcal{F}$ assembled by the leaves γ representing the arc β.

Let us consider the slice of $\mathcal{F}$ on U, that is, let $\mathcal{H} = \mathcal{F} \cap U$ and $X = W_{\mathcal{H}}$. To any leaf γ of $\mathcal{F}$, let us associate its itinerary $I(\gamma) = (\alpha_j(\gamma))$ on U. Let $\mathcal{I}(\beta)$ stand for the set of all non-trivial itineraries $\mathbf{a} = I(\gamma)$ corresponding to all possible leaves γ of $\mathcal{F}(\beta)$. By definition, $\mathbf{a} \longrightarrow \beta$ for any $\mathbf{a} \in \mathcal{I}(\beta)$. Let $\mathcal{F}(\beta, \mathbf{a})$ stand for the sublamination of $\mathcal{F}(\beta)$ assembled by the leaves γ with itinerary $\mathbf{a}$, i.e., $I(\gamma) = \mathbf{a}$.

For $\mathbf{a} = (\alpha_j) \in \mathcal{I}(\beta)$, let $v(\beta, \mathbf{a}) = W(\mathcal{F}(\beta, \mathbf{a}))$, and let $w_j(\beta, \mathbf{a})$ be the width of the lamination assembled by the segments of $\mathcal{F}(\beta, \mathbf{a}) \cap U$ corresponding to α_j. By the Series Law,

$$\bigoplus_j w_j(\beta, \mathbf{a}) \geq v(\beta, \mathbf{a}).$$

Moreover,

$$X = \sum_{\beta} \sum_{\mathbf{a}\in\mathcal{I}(\beta)} \sum_{j} w_j(\beta,\mathbf{a})\alpha_j, \quad Y = \sum_{\beta} \sum_{\mathbf{a}\in\mathcal{I}(\beta)} v(\beta,\mathbf{a})\beta.$$

This would mean that $X \multimap Y$ if we knew that $\mathcal{I}(\beta)$ were finite. The rest of the argument will show that the terms can be grouped so that the decompositions become finite.

Let $T(\beta)$ be the set of all IWAD's $\boldsymbol{\alpha} = \sum \alpha_j$ corresponding to all itineraries $\mathbf{a} = (\alpha_j) \in \mathcal{I}(\gamma)$, and let $\eta: \mathcal{I}(\beta) \to T(\beta)$ be the corresponding abelianization projection. Let $(e_i)_{i=1}^n$ be the maximal family of different non-trivial arcs represented by the leaves of $\mathcal{F} \cap U$. Then any IWAD $\boldsymbol{\alpha} \in T(\beta)$ has a form $\sum n_i e_i$, $n_i \in \mathbb{Z}_+$, and hence represents an element of the free semi-group $(\mathbb{Z}_+)^n$. By Lemma 3.6 below, there is a finite set $B(\beta) \subset T(\beta)$ such that any $\boldsymbol{\alpha} \in T(\beta)$ is greater or equal than some $\mathbf{a}' \in B(\beta)$. Let us show that the labeling set $\mathcal{I}(\beta)$ can be replaced with $B(\beta)$.

Let us select a projection $\pi: \mathcal{I}(\beta) \to B(\beta)$ factored through η and such that $\mathbf{a}' \equiv \pi(\mathbf{a}) \le \eta(\mathbf{a})$. Then the components of $\mathbf{a}' = \sum \alpha'_k$ is a subset of components of $\mathbf{a} = (\alpha_j)$, so that, we can select an injective map $j(k) = j_{\mathbf{a}}(k)$ such that $\alpha'_k = \alpha_{j(k)}$. Let

$$w'_k(\beta,\mathbf{a}') = \sum_{\pi(\mathbf{a})=\mathbf{a}'} w_{j_{\mathbf{a}}(k)}(\beta,\mathbf{a}), \quad v'(\beta,\mathbf{a}') = \sum_{\pi(\mathbf{a})=\mathbf{a}'} v(\beta,\mathbf{a}).$$

Then

$$X \ge \sum_{\beta} \sum_{\mathbf{a}'\in B(\beta)} \sum_{k} w'_k(\beta,\mathbf{a}')\,\alpha'_k, \quad Y = \sum_{\beta} \sum_{\mathbf{a}'\in B(\beta)} v'(\beta,\mathbf{a}')\beta.$$

Moreover, by Lemma 11.7

$$\bigoplus_k w'_k(\beta,\mathbf{a}') \ge v'(\beta,\mathbf{a}').$$

Thus, $X \multimap Y$. $\square$

Let us consider a free Abelian semigroup $S = \mathbb{Z}_+^n$ with the standard basis $(e_i)_{i=1}^n$. It is ordered coordinatewise: $\sum x_i e_i \ge \sum y_i e_i$ if $x_i \ge y_i$ for all i. For any $x \in S$, let $S_x = \{y \in S :\ y \ge x\}$ stand for the cone with the vertex at x.

Lemma 3.6. *Let T be an arbitrary subset of the semi-group S. Then there exist a finite subset $B \subset T$ such that $T \subset \bigcup_{x\in B} S_x$.*

Proof. Suppose first that T is non-empty. Let $a = \sum a_i e_i$ be an element of T. Let $\pi_i: \mathbb{Z}_+^n \to \mathbb{Z}$ be the projection onto the i^{th} coordinate. For $i \le n$, and $k < a_i$, $k \in \mathbb{Z}_+$, let $T_k^i = \pi_i^{-1}(\{k\}) \cap T$. Then, by induction on n, there is a finite $B_k^i \subset T_k^i$ such that

$$T_k^i \subset \bigcup_{x\in B_k^i} S_x.$$

Then let

$$B = \{a\} \cup \bigcup_{1\le i\le n} \bigcup_{k<a_i} B_k^i;$$

it has the desired property. $\square$

We can now prove Property D of canonical WAD's:

Lemma 3.7. *Let $U \subset V$. Then there exists a WAD $B \in W(U)$ with $\|B\|_\infty \leq 2$ such that*

$$W_{\text{can}}(U) + B \multimap W_{\text{can}}(V).$$

Proof. Since $W_{\text{can}}(V)$ is valid, Lemma 3.5 gives us a WAD X on U such that $X \multimap W_{\text{can}}(Y)$. By the Maximality Property, $X \leq W_{\text{can}}(U) + 2$. Hence there exists a WAD B with $\operatorname{supp} B \subset \operatorname{supp} X$, $\|B\|_\infty \leq 2$, and $X \leq W_{\text{can}}(U) + B$. The conclusion follows. □

Lemma 3.8. *If*

$$\bigoplus (x_i + b_i) \geq y,$$

then

$$\bigoplus x_i \geq y - \sum b_i.$$

Proof.

$$\frac{\partial}{\partial x_i} \bigoplus x_k = \frac{(\bigoplus x_k)^2}{x_i^2} \leq 1.$$

Therefore

$$\bigoplus (x_i + b_i) \leq \bigoplus x_i \; + \; \sum b_i.$$

□

Lemma 3.9. *If $X + B \multimap Y$, then $X \multimap Y - \|B_1\|_1$.*

Proof. Now suppose $X + B \multimap Y$. Formally we can write $X + B \geq \sum_i T_i$, $Y = \sum Y_i$, where $T_i = \sum_j w_{ij} \alpha_{ij}$, $Y_i = v_i \beta_i$, and

$$(\alpha_{ij})_j \to \beta_i, \tag{3.2}$$

and

$$\bigoplus_j w_{ij} > v_i.$$

By the general theory of positive vectors in $\mathbb{R}^n$, we can write $T_i = X_i + B_i$, where $X \geq \sum X_i$, and $B \geq \sum B_i$. So writing $X_i = \sum_j w_{ij}^X \alpha_{ij}$, and $B_i = \sum_j w_{ij}^B \alpha_{ij}$, we obtain

$$\bigoplus_j (w_{ij}^X + w_{ij}^B) \geq v_i,$$

so, by Lemma 3.8,

$$\bigoplus_j w_{ij}^X \geq v_i - \|B_i\|_1,$$

and therefore (using (3.2))

$$X \geq \sum_i X_i \; \multimap \; \sum_i (v_i - \|B_i\|_1) \beta_i \; \geq \; Y - \|B\|_1.$$

□

Together with Lemma 3.7, this implies:

Corollary 3.10. *Let $U \subset V$. Then*

$$W_{\text{can}}(U) \multimap W_{\text{can}}(V) - 6|\chi(U)|.$$

3.7. **Horizontal and vertical arc diagrams.** In the further applications, the Riemann surface S will be $U \setminus \mathcal{K}$, where U is a topological disk, and $\mathcal{K}$ is the union of p disjoint full continua (in reality, the cycle of little Julia sets of some quadratic-like map).

Under these circumstances, a proper path (and the corresponding arc) in $U \setminus \mathcal{K}$ is called *horizontal* if it connects two little Julia sets, and is called *vertical* if it connects a little Julia set to ∂U. Given an arc diagram W on $U \setminus \mathcal{K}$, let W^{h} and W^{v} stand for its horizontal and vertical parts respectively, and let $W^{\mathrm{v+h}}$ stand for their sum. In particular, we can consider $W^{\mathrm{h}}_{\mathrm{can}}$, $W^{\mathrm{v}}_{\mathrm{can}}$ and $W^{\mathrm{v+h}}$.

4. Life on Hubbard trees

4.1. **Topological disked trees and aligned graphs.** Let us consider a Riemann surface U with finitely many open Jordan disks $D_j \Subset U$ with disjoint closures, and let $\mathcal{D} = \cup D_j$. An embedded 1-complex $H_k \subset U \setminus \mathcal{D}$ is called *proper* if all its tips (i.e., valence 1 vertices) belong to $\partial \mathcal{D}$. Assume that we have finitely many disjoint proper graphs H_k such that $H = \mathcal{D} \cup_k H_k$ is simply connected. In this case we say that H is a *(topological) disked tree*.

We say that a path in $H \setminus \mathcal{D}$ is *aligned* with a disked tree H if it connects the boundaries of two disks in $\mathcal{D}$. The arcs α in $U \setminus \mathcal{D}$ represented by aligned paths are also called *aligned with* H. Let $\mathbf{H}$ stand for the family of the aligned paths/arcs. (Since there is a natural one-to-one correspondence between the aligned arcs and the aligned paths, we will not distinguish notationally these families).

Let $\mathbf{G}$ be the abstract graph whose vertices are disks D_i and edges are the paths/arcs of $\mathbf{H}$. We call it the *graph of aligned arcs*.

Lemma 4.1. *The graph* $\mathbf{G}$ *is a tree of complete graphs.*[9]

Proof. Let $\mathcal{D}_k$ stand for the family of the disks D_i that touch the graph H_k. Let $\mathbf{G}_k \subset G$ be the graph whose vertices are disks $D_i \subset \mathcal{D}_k$ and edges are the path of $\mathbf{H}$ contained in $\mathbf{G}_k$. As H_k is path connected, any two disks D_i and D_j of $\mathcal{D}_k$ can be connected by an arc in $\mathbf{G}_k$, so that, the graph $\mathbf{G}_k$ is full.

Since H is a tree, it is easy to check that the graphs $\mathbf{G}_k$ are organized in a tree as well. $\square$

4.2. **Superattracting model.** To a primitively p-renormalizable map f, one can associate a certain superattracting quadratic polynomial F of period p as follows (compare [McM1, §B]). First, we can straighten f to a quadratic polynomial, so we can assume that it is a quadratic polynomial in the first place. Then, collapsing the little Julia sets K_j to points c_j, we obtain a topological sphere S. Moreover, the map f descends to a degree two map $f_0 \colon S \to S$ with a periodic critical point c_0 (f_0 collapses the sets $-K_j \subset \mathbb{C} \setminus \mathcal{K}$, $j = 1, \dots, p-1$), to points c_{j+1}. One can check that this map does not have Thurston obstructions (see [DH3]), so it can be realized as a superattracting quadratic polynomial. This polynomial is the desired F.

We call F the *superattracting model* for f. We let $\mathcal{O} = \{F^k(0)\}_{k=0}^{p-1}$ stand for the superattracting cycle of F, and $\mathcal{D}$ stand for its immediate basin of attraction.

Lemma 4.2. *Let* $f \colon (U, \mathcal{K}) \to (V, \mathcal{K})$ *be a primitively renormalizable quadratic-like map, and let* $F \colon (\mathbb{C}, \mathcal{D}) \to (\mathbb{C}, \mathcal{D})$ *be its superattracting model. There is a natural one-to-one correspondence between horizontal/vertical arcs in* $U \setminus \mathcal{K}$ *(resp.,* $V \setminus f^{-1}(\mathcal{K})$*) and horizontal/vertical*

[9]See §11.7 of Appendix B for the definition.

arcs in $\mathbb{C} \smallsetminus \mathcal{D}$ (resp. $\mathbb{C} \smallsetminus F^{-1}(\mathcal{D})$). The correspondence between the horizontal arcs is compatible with the arrow relation.

This obvious statement allows us to replace f with its superattracting model F, as long as we are dealing with the combinatorics of arc diagrams. The advantage of the model is that it possesses a well-defined Hubbard tree.

4.3. **Disked Hubbard tree.** Let D_k be the component of $\mathcal{D}$ containing c_k. It is known that the $\bar{D}_k$'s are Jordan disks. Given a set $X \subset K(F)$, the *path hull* of X is defined as the smallest path connected closed subset of $K(F)$ containing X satisfying the property that it intersects any component of $\operatorname{int} K(F)$ by the union of internal rays. The *Hubbard tree* $H \equiv H_F$ is defined as the path hull of the basin $\mathcal{D}$. It is a disked tree in the sense of §4.1. Moreover, it is invariant under F; in fact, $F(H) = H$.

Lemma 4.3. *The valence of any disk D_k of the Hubbard tree is at most 2.*

4.4. **No periodic horizontal arcs.** Given a proper path γ in $\mathbb{C} \smallsetminus \mathcal{D}$ that begins at some D_i, $i \in \mathbb{Z}/p\mathbb{Z}$, let $F^*\gamma$ stand for the union of the proper lifts of γ that begin at D_{i-1}. (Note that a horizontal path γ has at most one proper lift.)

This lifting operation descends to the level of arcs. An arc **a** is called *periodic* if $(F^*)^l(\alpha) \supset \alpha$ for some $l \in \mathbb{Z}_+$. (Of course, the period l must be a multiple of p.)

Lemma 4.4 (see [P], Theorem 5.8)**.** *Periodic horizontal arcs do not exist.*

Proof. Assume that α is such a periodic horizontal arc with period l. Let us endow $\mathbb{C} \smallsetminus \mathcal{O}$ with hyperbolic metric, and let us represent α be a geodesic path $\gamma \subset \mathbb{C} \smallsetminus \mathcal{D}$. It is the shortest representative of α.

Since $(F^*)^l\alpha = \alpha$, the lift $F^*\gamma$ represents the same arc α. so that, it is no shorter than γ. On the other hand, the Schwarz Lemma implies that it is shorter than γ – contradiction. ☐

4.5. **Expansion property.** Let $F^{-1}\mathbf{H}$ stand for the family of paths in $H \smallsetminus F^{-1}\mathcal{D}$ with endpoints on $F^{-1}\bar{\mathcal{D}}$. Each path of $F^{-1}\mathbf{H}$ is homeomorphically mapped by F onto some path of $\mathbf{H}$.

Let us consider a $|\mathbf{H}| \times |\mathbf{H}|$ matrix $M = M(F)$ defined as follows: $M_{\gamma\delta} = n$ if γ contains n disjoint segments $\gamma_i \in F^{-1}\mathbf{H}$ such that $F\gamma_i = \delta$. One can easily check that the matrix M^k can obtained by applying the same construction to the map F^k. .

Lemma 4.5. *For any $\gamma \in \mathbf{H}$, we have $\displaystyle\sum_\delta M^p_{\gamma\delta} \geq 2$.*

Proof. Not that the path $F^p\gamma$ has the same endpoints as γ does. If $\displaystyle\sum_\delta M^p_{\gamma\delta} = 1$ then $F^p\gamma$ would not cross any disks of $\mathcal{D}$. It would follow that $f^p\gamma = \gamma$, which is impossible by Lemma 4.4. ☐

4.6. **Periodic vertical arcs.** Let $\mathbf{H}^\perp$ be the arc diagram consisting of vertical arcs in $\mathbb{C} \smallsetminus \mathcal{D}$ that do not intersect the Hubbard tree H (up to homotopy).

Lemma 4.6. *If a vertical arc β is periodic then $\beta \in \mathbf{H}^\perp$.*

Proof. Let $(F^*)^l\beta \supset \beta$ for some $l \in \mathbb{Z}_+$. Let $\mathcal{G}$ be the family of paths γ in $\mathbb{C} \smallsetminus \mathcal{D}$ representing β. For a path $\gamma \in \mathcal{G}$, let γ' be its maximal initial segment whose endpoints belong to the Julia set $K(F)$. Let $\mathcal{G}'$ be the family of all paths γ' that can be obtained in this way.

Since $(F^*)^l\beta \supset \beta$, any path $\gamma \in \mathcal{G}$ can be lifted by F^l to a path $\delta \in \mathcal{G}$. Then δ' is a lift of γ', and thus, we obtain the lifting map $L\colon \mathcal{G}' \to \mathcal{G}'$.

Let us endow the punctured plane $S = \mathbb{C} \smallsetminus \mathcal{O}$ with the hyperbolic metric. Let $|\gamma|$ stand for the hyperbolic length of a path γ, and let

$$\mu = \inf_{\gamma' \in \mathcal{G}'} |\gamma'|.$$

This infimum is realized by some hyperbolic geodesic γ'. If $\mu > 0$ then by the Schwarz Lemma, $|L(\gamma')| < |\gamma'|$, contradicting minimality of γ'. Hence $\mu = 0$ and so, $\mathcal{G}$ contains a curve γ that does not intersect the Julia set $K(F)$ (except for the initial point in $\mathcal{D}$). This is the desired vertical representative of β. $\square$

4.7. **Pullbacks of vertical arcs.** Given two families $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ of proper arcs (or paths) in a Riemann surface S, the "inner product" $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle \equiv \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle_S$ is the minimal number of intersection points between path representatives of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

For instance, $\alpha \in \mathbf{H}^\perp$ if and only if $\langle \alpha, \mathbf{H} \rangle = 0$. The *dual* statement is also valid: $\alpha \in \mathbf{H}$ if and only if $\langle \alpha, \mathbf{H}^\perp \rangle = 0$.

Lemma 4.7. *For any vertical arc β, we have:* $\displaystyle\bigcup_{n=0}^{\infty} (F^n)^*(\beta) \supset \mathbf{H}^\perp$.

Proof. Let γ be a vertical path representing β, and let γ_1 be a component of $F^*\gamma$. Since $F\colon \gamma_1 \to \gamma$ is a homeomorphism,

$$\langle \gamma_1, F^{-1}\mathbf{H} \rangle_{\mathbb{C} \smallsetminus F^{-1}\mathcal{D}} \;=\; \langle \gamma, \mathbf{H} \rangle_{\mathbb{C} \smallsetminus \mathcal{D}}.$$

Since $H \subset F^{-1}H$,

$$\langle \gamma_1, \mathbf{H} \rangle \;\le\; \langle \gamma, \mathbf{H} \rangle. \tag{4.1}$$

Let us now consider a chain of vertical arcs $\beta_n \subset (F^n)^*(\beta)$. By (4.1), the intersection number $\langle \beta_n, \mathbf{H} \rangle$ does not increase with n, and hence eventually stabilizes at some value k. But for a given k, there are only finitely many different vertical arcs β such that $\langle \beta, \mathbf{H} \rangle = k$. Hence $\beta_n = \beta_{n+l}$ for some $l > 0$. By Lemma 4.6, $\beta_n \in \mathbf{H}^\perp$.

But then all further lifts of β_n by the iterates of F belong to $\mathbf{H}^\perp$ as well. Lifting β_n to the critical disk D_0, we obtain two symmetric arcs, σ_1 and σ_2, landing at D_0. By symmetry, these arcs are different. But by Lemma 4.3, for any disk D_k, there exist at most two different arcs of $\mathbf{H}^\perp$ landing on D_k. Thus, σ_1 and σ_2 are the only arcs of $\mathbf{H}^\perp$ landing on D_0. Lifting these arcs further to all the domains D_k, we obtain all the arcs of $\mathbf{H}^\perp$. $\square$

4.8. **Trees H^l and the associated objects.** For any $l \in \mathbb{Z}_\ge$, we can introduce the following objects:

- $\mathcal{D}^l = F^{-l}(\mathcal{D})$; D^l_k are the components of $\mathcal{D}$;
- $H^l = F^{-l}(H)$; it is a disked tree with disks D^l_k;
- $\mathbf{H}^l$ is the family of the paths/arcs aligned with H^l;
- $\mathbf{G}^l$ is a graph whose vertices are the disks D^l_k and edges are the arcs of $\mathbf{H}^l$; as for $l = 0$ (Lemma 4.1), it is a tree of complete graphs $\mathbf{G}^l_k$.

Notice that F^l maps $\mathbf{G}^l$ onto $\mathbf{G}$, so that each complete graph $\mathbf{G}^l_k$ is mapped onto some complete graph $\mathbf{G}_j$.

By Lemma 11.10 applied to $\mathbf{G}^l$, for any edge $[D_m, D_n]$ of $\mathbf{H}$, there is a unique chain of $\mathbf{G}^l$,

$$D_m = D^l_{k(1)}, D^l_{k(2)}, \dots, D^l_{k(d)} = D_n, \tag{4.2}$$

connecting D_m to D_n. Here $d = d_{\mathbf{G}^l}(D_m, D_n)$ is the distance between D_m and D_n in $\mathbf{G}^l$. Lemma 4.5 implies:

$$d_{\mathbf{G}^{rp}}(D_m, D_n) \geq 2^r. \tag{4.3}$$

In fact, we have:

Lemma 4.8. *At least 2^{r-1} of the disks $D^{rp}_{k(i)}$ belong to $\mathcal{D}^{rp} \smallsetminus \mathcal{D}^{(r-1)p}$.*

Proof. For $r = 1$, the assertion follows from Lemma 4.5 and a remark that all the intermediate disks in the path (4.2) do not belong to $\mathcal{D}$. Since each path of $\mathbf{H}^p$ is mapped homeomorphically under F^p onto a path of $\mathbf{H}$ (transferring chains in $\mathbf{G}^{rp}$ to chains in $\mathbf{G}^{(r-1)p}$), the assertion follows by induction in r. □

5. Restrictions of WAD's

5.1. **Restriction of the domains.** Let us consider a ψ-quadratic-like map $\mathbf{f} = (i, f)\colon \mathbf{U}^1 \to \mathbf{U}^0$, and let $\mathbf{f}^n = (i_n, f_n)\colon \mathbf{U}^n \to \mathbf{U}^0$ be its n-fold iterate. Then there is a natural ψ-quadratic-like map $\mathbf{U}^{n+1} \to \mathbf{U}^n$ called the *restriction* of $\mathbf{f}$ to $\mathbf{U}^{n+1}$. We will use the same notation $\mathbf{f} = (i, f)$ for these restrictions.

Assume f is primitively renormalizable with some period p, and let $\mathcal{K}$ be the cycle of the corresponding little (filled) Julia sets. Let $\tilde{\mathcal{K}} = f^{-1}(\mathcal{K})$. Then we have an immersion $i\colon \mathbf{U}^{n+1} \smallsetminus \mathcal{K} \to \mathbf{U}^n \smallsetminus \mathcal{K}$, a covering $f\colon \mathbf{U}^{n+1} \smallsetminus \tilde{\mathcal{K}} \to \mathbf{U}^n \smallsetminus \mathcal{K}$, and an embedding $\mathbf{U}^{n+1} \smallsetminus \tilde{\mathcal{K}} \subset \mathbf{U}^{n+1} \smallsetminus \mathcal{K}$, that together form a triangle diagram. We will properly mark the Riemann surfaces $\mathbf{U}^n \smallsetminus \mathcal{K}$ and $\mathbf{U}^n \smallsetminus \tilde{\mathcal{K}}$ by declaring the ends corresponding to the little Julia sets K_i to be proper. Then the maps in the above diagram are partially proper. Moreover, the embedding $\mathbf{U}^{n+1} \smallsetminus \tilde{\mathcal{K}} \subset \mathbf{U}^{n+1} \smallsetminus \mathcal{K}$ is also proper on $\partial \mathbf{U}^{n+1}$.

5.2. **Increase of the total weight.**

Lemma 5.1. *For any $n \in \mathbb{Z}_{\geq 0}$, we have:*

$$\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1 \leq \|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U}^n \smallsetminus \mathcal{K})\|_1 + 6(p+1).$$

Proof. Since the immersion $i_n\colon \mathbf{U}^n \to \mathbf{U}$ is proper on $\mathcal{K}$, any proper path in $\mathbf{U} \smallsetminus \mathcal{K}$ beginning on K_j contains an sub-path that lifts to a proper path in $\mathbf{U}^n$ beginning on the same K_j. (Moreover, vertical path lift to vertical ones, while horizontal paths may lift to either horizontal or vertical.) Hence the canonical foliation $\mathcal{F} \equiv \mathcal{F}^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})$ lifts to a proper foliation $\mathcal{F}^n$ on $\mathbf{U}^n \smallsetminus \mathcal{K}$. Hence

$$\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1 = \mathcal{W}(\mathcal{F}) \leq \mathcal{W}(\mathcal{F}^n) \leq \|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U}^n \smallsetminus \mathcal{K})\|_1 + 6(p+1),$$

where the first estimate follows from Corollary 10.2, while the last one follows from Property A. □

5.3. **Domination relations.** Let us select a non-decreasing sequence of numbers $q_n \in \mathbb{R}_+$ such that

$$q_{n+1} \geq 3p(q_n + 2). \tag{5.1}$$

Let $X^n = W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U}^n \smallsetminus \mathcal{K})$ and $\hat{X}^n = X^n - q_n$.

Proposition 5.2. *We have:*

(i) $\hat{X}^{n+1} \leq i^* \hat{X}^n$;

(ii) $f^* \hat{X}^n \multimap \hat{X}^{n+1}$.

Proof. (i) Since the immersion i is proper on the ends corresponding to the little Julia sets K_j, we have $X^{n+1} \leq i^* X^n$ by Property C. This yields the desired inequality since the sequence (q_n) is non-decreasing.

(ii) Let us properly mark the ends of $\mathbf{U}^n \smallsetminus \mathcal{K}$ and $\mathbf{U}^{n+1} \smallsetminus \tilde{\mathcal{K}}$ corresponding to the sets $\mathcal{K}$ and $\tilde{\mathcal{K}}$ respectively. Since the covering f maps $\partial \mathbf{U}^{n+1}$ to $\partial \mathbf{U}^n$ and maps $\tilde{\mathcal{K}}$ to $\mathcal{K}$, horizontal and vertical arcs of $\mathbf{U}^n \smallsetminus \mathcal{K}$ lift respectively to horizontal and vertical arcs of $\mathbf{U}^{n+1} \smallsetminus \tilde{\mathcal{K}}$. Hence $f^* X^n = W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U}^{n+1} \smallsetminus \tilde{\mathcal{K}})$.

By Property D,

$$W_{\mathrm{can}}(\mathbf{U}^{n+1} \smallsetminus \tilde{\mathcal{K}}) \multimap W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U}^{n+1} \smallsetminus \mathcal{K}) - 6p.$$

But since the embedding $\mathbf{U}^{n+1} \smallsetminus \tilde{\mathcal{K}} \subset \mathbf{U}^{n+1} \smallsetminus \mathcal{K}$ is proper on $\partial \mathbf{U}^{n+1}$, the itinerary of any horizontal path γ in $\mathbf{U}^{n+1} \smallsetminus \mathcal{K}$ consists only of horizontal arcs of $\mathbf{U}^{n+1} \smallsetminus \tilde{\mathcal{K}}$. It follows that

$$W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U}^{n+1} \smallsetminus \tilde{\mathcal{K}}) \multimap W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U}^{n+1} \smallsetminus \mathcal{K}) - 6p.$$

Thus, $f^* X^n \multimap X^{n+1} - 6p$. Taking into account Lemma 3.8 and (5.1), we conclude:

$$f^* \hat{X}^n = f^* X^n - q_n \multimap X^{n+1} - 6p - 3pq_n \geq \hat{X}^{n+1}.$$

□

5.4. **Topological arrow.** Let us consider two Riemann surfaces, $U \subset V$, and let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be multiarcs on U and V respectively. We say that $\boldsymbol{\alpha}$ *topologically arrows* $\boldsymbol{\beta}$, $\boldsymbol{\alpha} \rightsquigarrow \boldsymbol{\beta}$, if for any arc $\beta \in \boldsymbol{\beta}$ there is a sequence (α_k) of arcs with $\alpha_k \in \boldsymbol{\alpha}$ for each k, and $(\alpha_k) \longrightarrow \beta$.

A basic example comes from two WAD's, $X \in W(U)$ and $Y \in W(V)$, such that $X \multimap Y$. Then $\operatorname{supp} X \rightsquigarrow \operatorname{supp} Y$, as follows immediately from the definitions.

5.5. **Invariant horizontal arc diagram.** Given a partially proper embedding $U \subset V$ such that U is a deformation retract of V, we can view horizontal arcs on V as horizontal arcs on U (by retracting them from V to U). When we'd like to emphasize this point of view, we will use notation $\alpha | U$.

Let us say that a horizontal arc diagram $\boldsymbol{\alpha}$ on $\mathbf{U}^n \smallsetminus \mathcal{K}$ is *invariant* if

$$f^* \boldsymbol{\alpha} \rightsquigarrow \boldsymbol{\alpha}$$

Proposition 5.3. *There exists $n \leq 3p$ such that the horizontal arc diagram $\boldsymbol{\alpha}^n = \operatorname{supp} \hat{X}^n$ is invariant.*

Proof. Since $\hat{X}^{n+1} \leq i^* \hat{X}^n$ (by Lemma 5.2 (i)), $\boldsymbol{\alpha}^{n+1} \subset \boldsymbol{\alpha}^n | \mathbf{U}^{n+1} \smallsetminus \mathcal{K}$. Since $|\boldsymbol{\alpha}^0| \leq 3p$, there exists an $n \leq 3p$ such that $\boldsymbol{\alpha}^{n+1} = \boldsymbol{\alpha}^n | \mathbf{U}^{n+1} \smallsetminus \mathcal{K}$. Since by Lemma 5.2 (ii), $f^* \boldsymbol{\alpha}^n \rightsquigarrow \boldsymbol{\alpha}^{n+1}$, we are done. □

5.6. **Alignment with the Hubbard tree.** We say that a horizontal arc diagram $\mathbf{a}$ is *aligned with the Hubbard tree* if it is so for the superattracting realization of $\mathbf{a}$.

Lemma 5.4. *Any invariant horizontal arc diagram $\boldsymbol{\alpha}$ is aligned with the Hubbard tree.*

Proof. Let $\boldsymbol{\alpha}$ be an invariant horizontal arc diagram for a superattracting polynomial F. It can be realized by a "disked graph" A whose "vertices" are the disks $\bar{D}_k$ and edges are paths representing the arcs of $\boldsymbol{\alpha}$. Let Δ be the unbounded component of $\mathbb{C} \smallsetminus A$. Then one of the disks $\bar{D}_k$ intersects $\partial\Delta$. A path γ in Δ landing on $\bar{D}_k$ represents a vertical arc β such that $\langle \boldsymbol{\alpha}, \beta \rangle = 0$.

Then $F^*\gamma$ does not intersect $F^{-1}(A)$ (except for the landing points). But by invariance, the arcs of $\boldsymbol{\alpha}$ can be realized as paths in $F^{-1}(A)$. Hence $F^*\gamma$ does not intersect $\boldsymbol{\alpha}$.

Iterating, we see that $\boldsymbol{\alpha}$ does not intersect $(F^n)^*\gamma$ for all $n = 0, 1, \dots$. By Lemma 4.7, it does not intersect $\mathbf{H}^\perp$ either. So, $\langle \boldsymbol{\alpha}, \mathbf{H}^\perp \rangle = 0$, and we are done. ☐

Putting this together with Proposition 5.3 and Proposition 5.2 (i), we obtain:

Corollary 5.5. *For any $n \geq 3p$, the horizontal arc diagram $\boldsymbol{\alpha}^n = \operatorname{supp} \hat{X}^n$ is aligned with the Hubbard tree H.*

6. Entropy argument

6.1. **Domination and electric circuits.** Let us consider two topological disked trees $H \subset H'$, with the families of disks $\{D_j\} \subset \{D'_i\}$. Then the corresponding aligned arc diagrams are related by the topological arrow: $\mathbf{H'} \rightsquigarrow \mathbf{H}$. Let $\mathbf{G}$ and $\mathbf{G'}$ be the corresponding trees of complete graphs.

We say that WAD Y is aligned with H if $\operatorname{supp} Y$ is aligned with H. Such a diagram induces an unplugged electric circuit $\mathcal{C}_Y$ based on $\mathbf{G}$ by letting the conductance of the edge $e \in \mathbf{H}$ be $Y(e)$. The following lemma relates the domination relation between aligned WAD's to the domination relation between the corresponding electric circuits (see §11.6).

Lemma 6.1. *Let WAD's Y and Y' be aligned with trees $H \subset H'$ as above. If $Y' \multimap Y$ then $\mathcal{C}_{Y'} \multimap \mathcal{C}_Y$.*

Proof. By definition of domination, there exist edges $\beta_i \in \mathbf{H}$ concatenated by paths $(\alpha_{ij})_j$ in $\mathbf{H'}$, and positive numbers w_i, v_{ij} such that

$$Y = \sum_i w_i \beta_i, \quad Y' \geq \sum_i \sum_j v_{ij} \alpha_{ij}, \tag{6.1}$$

and for any i,

$$w_i \leq \bigoplus_j v_{ij}. \tag{6.2}$$

Let $B(e)$ be the family of edges β_i equal to e. For any $\beta_i \in B(e)$, let us consider an auxiliary electric circuit $\mathcal{C}'_i$ whose resistors are $(\alpha_{ij})_j$ with conductances $(v_{ij})_j$ plugged in series with battery ∂e. Then (6.2) translates into an estimate: $w_i \leq \mathbf{W}(\mathcal{C}'_i)$, where $\mathbf{W}(\mathcal{C}'_i) = \bigoplus_j v_{ij}$ is the conductance of $\mathcal{C}'_i$.

Each electric circuit $\mathcal{C}'_i$ admits a natural projection into the circuit $\mathcal{C}'_Y$. By Lemma 11.6 and the second inequality of (6.1), $\sum \mathbf{W}(\mathcal{C}_i) \le \mathbf{W}(\mathcal{C}_{Y'}(e))$. Hence

$$Y(e) = \sum_{\beta_i \in B(e)} w_i \le \sum_{\beta_i \in B(e)} \mathbf{W}(\mathcal{C}_i) \le \mathbf{W}(\mathcal{C}_{Y'}(e)),$$

and we are done. $\square$

6.2. **Dynamical setting.** Let us now consider the Hubbard tree H of a superattracting map F with the basin $\mathcal{D} = \cup D_j$, where $j \in \mathbb{Z}/pZ$. If Y is a WAD aligned with H, we let

$$Y|\, D_j = \sum_{\partial\alpha \ni D_j} Y(\alpha)$$

be the local conductances of the associated electric circuit (see §11.5).

Lemma 6.2. *If Y and Z are WAD's aligned with the Hubbard tree H and such that $F^*Y \multimap Z$, then for any two disk D_j we have:*

$$Z|\, D_j \le (\deg F|D_j) \cdot Y|\, D_{j+1}.$$

Proof. We have $F^*Y|\, D_j = \deg(F|D_j) \cdot Y|\, D_{j+1}$. By Lemma 6.1, $\mathcal{C}_{F*Y} \multimap \mathcal{C}_Y$. Hence $F^*Y|\, D_j \ge Z|\, D_j$ by Lemma 11.9. Thus,

$$Z|\, D_j \le (\deg F|D_j) \cdot Y|\, D_{j+1}.$$

$\square$

Lemma 6.3. *Let Y and Z be WAD's aligned with the Hubbard trees H. Assume $(F^{rp})^*Y \multimap Z$ for some $r \in \mathbb{Z}_+$ and some WAD Z. Then for any edge α of $\mathbf{H}$, we have:*

$$Z(\alpha) \le \frac{1}{2^{r-2}} \max_j (Y|D_j).$$

Proof. The WAD $(F^{rp})^*Y$ is aligned with the tree $H^{rp} = F^{-rp}H \supset H$ (see §4.8). Let $\mathcal{C}^l$ be the associated electric circuit. Recall that $\mathcal{C}^l(\alpha)$ stands for the restriction of this circuit to the connected component of $\mathbf{G}^l \setminus \mathcal{D}$ attached to $\partial\alpha$. By Lemma 6.1, $\mathcal{C}^l \multimap \mathcal{C}$, so that,

$$Z(\alpha) \le \mathbf{W}(\mathcal{C}^l(\alpha)).$$

Let us consider the path of disks $(D_k^{rp})_k$ connecting $\partial\alpha$ in the tree of graphs $\mathbf{G}^{rp}$ (see Lemma 11.10). By Lemma 11.11

$$\mathbf{W}(\mathcal{C}^l(\alpha)) \le \bigoplus_k (F^{rp})^*Y|\, D_k^{rp}.$$

If $D_k^{rp} \in \mathcal{D}^{rp} \setminus \mathcal{D}^{(r-1)p}$ then F^{rp} maps D_k^{rp} onto some disk D_j with degree at most 2. Hence

$$(F^{rp})^*Y|\, D_k^{rp} \le 2Y|\, D_j.$$

Since by Lemma 4.8, there are at least 2^{r-1} such disks, we have:

$$\bigoplus_k (F^{rp})^*Y|\, D_k^{rp} \le \frac{1}{2^{r-2}} \max_j (Y|\, D_j).$$

Putting the above estimates together, we obtain the desired. $\square$

6.3. **Loss of the horizontal weight.** Recall now the WAD's $\hat{X}^n = X^n - q_n$ from §5.3. By Lemma 4.2, they can be viewed as WAD's of the superattracting model F. We will keep the same notation for these diagrams.

Lemma 6.4. *Let p be the period of* 0 *under F. Then*

$$\|\hat{X}^{10p}\|_1 \leq \frac{1}{4}\|\hat{X}\|_1.$$

Proof. By Corollary 5.5, the WAD's $\hat{X}^n$ are aligned with the Hubbard tree H for $n \geq 3p$. By Proposition 5.2 (ii), $(F^{6p})^*\hat{X}^{4p} \multimap \hat{X}^{10p}$. Hence by Lemma 6.3,

$$\hat{X}^{10p}(\alpha) \leq \frac{1}{16}\max_j(\hat{X}^{4p}|\, D_j).$$

By Lemma 6.2, $(\hat{X}^{4p}|\, D_j) \leq 2(\hat{X}^{3p}|\, D_k)$ for any two disks D_j and D_k. Hence

$$\max(\hat{X}^{4p}|\, D_j) \leq \frac{2}{p}\|\hat{X}^{3p}\|_1 \leq \frac{2}{p}\|\hat{X}|\|_1,$$

where the last estimate comes from Proposition 5.2(i).

Putting the above estimates together and summing it up over all $\alpha \in \operatorname{supp} X^{10p}$ (taking into account that $\operatorname{supp} X^{10p}$ contains at most at most $3p$ arcs), we obtain:

$$\|\hat{X}^{10p}\|_1 \leq \frac{3}{8}\|\hat{X}|\, D_j\|_1.$$

□

Let us now go back to the original map f. The following result shows that after an appropriate restriction of the domain of f, there is a definite loss of the horizontal weight of the associated canonical WAD.

Corollary 6.5. *Let f be a renormalizable ψ-ql map with period p. Then there exists $M = M(p)$ such that*

$$\|W^h_{\mathrm{can}}(\mathbf{U}^{mp} \smallsetminus \mathcal{K})\|_1 \leq \frac{1}{2}\|W^h_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1,$$

provided $\|W^h_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1 > M(p)$.

Proof. The last lemma immediately yields

$$\|W^h_{\mathrm{can}}(\mathbf{U}^{10p} \smallsetminus \mathcal{K})\|_1 \leq \frac{1}{4}\|W^h_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1 + 3pq_{10p},$$

which implies the desired. □

Combining this with Lemma 5.1, we obtain:

Corollary 6.6. *Let f be a renormalizable ψ-ql map with period p. Then there exists $M = M(p)$ such that*

$$\|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U}^{10p} \smallsetminus \mathcal{K})\|_1 \geq \frac{1}{2}\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1,$$

provided $\|W^h_{\mathrm{can}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1 > M(p)$.

7. Push-forward Argument

7.1. **Quasi-Additivity Law.** In this section we will formulate the main result of [KL1].

Let $\beta_i(U)$ stand for the Betti numbers of a Riemann surface U. We call the number $(\beta_0+\beta_1)(U)$ the *topological complexity* of U.

We say that a compact subset K in a Riemann surface S has a *finite type* if it is the intersection of a nest of compact Riemann subsurfaces (with boundary) U_i of bounded complexity. Connected compact sets of finite type will be also called *islands*.

Let A_j $(j=1,\dots,N)$ be a finite family of disjoint islands in S. There are 3 conformal moduli associated with such a family:

$$X = \mathcal{W}(S, \bigcup_{j=1}^{N} A_j);$$

$$Y = \sum_{j=1}^{N} \mathcal{W}(S, A_j), \tag{7.1}$$

$$Z = \sum_{j=1}^{N} \mathcal{W}(S \smallsetminus \bigcup_{k\neq j} A_k,\ A_j).$$

It is easy to see that $X \le Y \le Z$. We say that the islands A_j are *ξ-separated* if $Z \le \xi Y$. The following Quasi-Additivity Law tells us that in a near degenerate situation (when Y is big), under the separation assumption, the moduli X and Y are comparable:

Quasi-Additivity Law. [KL1]. *Assume that the islands $A_j \Subset \operatorname{int} S$ are ξ-separated. Then there exists M depending only on ξ and the topological complexity of the family of islands such that:*
If $Y \ge M$ then $Y \le C\xi X$, where C is an absolute constant.

7.2. **Covering Lemma.** We will now give a version of the Covering Lemma of [KL1] suitable for our purposes.

Covering Lemma. *For any natural numbers $p, D \ge d$ and any $\xi > 0$, there exists a $L = L(p, D, \eta)$ with the following property. Let U and U' be two topological disks, and let $(K_j)_{j=1}^p$ and $(K'_j)_{j=1}^p$ be two families of disjoint FJ-setsets in U and U' respectively. Let $f\colon (U, \cup K_j) \to (U', \cup K'_j)$ be a branched covering with critical values in $\cup K'_j$ such that $\deg(f\colon U \to U') = D$, K_j is a component of $f^{-1}(K'_j)$, and $\deg(f\colon K_j \to K'_j) \le d$, $j = 1,\dots,p$. Let (X, Y, Z) and (X', Y', Z') stand for the associated conformal moduli. Assume*

$$\|W^{\mathrm{v+h}}(U' \smallsetminus \cup K'_j)\|_1 \le \xi Y$$

(the collar assumption). If $Y > L$ then

$$X \le C\xi d^2 X',$$

where C is an absolute constant.

Proof. Let $E \subset U'$ be the set of critical values of f, Then there exists a Galois branched covering $g\colon S \to U'$ of degree at most $D!$ with critical values in E that factors through f,

i.e., $g = f \circ h$, where $h\colon S \to U$ is also a Galois branched covering (see e.g., Proposition 3.1 of [KL1]). Let Γ be the group of deck transformations of the covering g.

Let $A_j^i \subset S$ be the connected components of $g^{-1}(K_j')$, $j = 1, \ldots, p$, labeled in such a way that $A_j \equiv A_j^1$ satisfies: $h(A_j^1) = K_j$. For a given j, these components are transitively permuted by Γ.

We now let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be the moduli associated with the family of isles A_j^i in S. By Lemma 10.5 from the Appendix, we have:

$$\mathcal{X} = |\Gamma|\, X'. \tag{7.2}$$

Let $d_j = \deg(f\colon K_j \to K_j')$, $m_j = \deg(h\colon A_j \to K_j)$. Then the stabilizer of A_j consists of $d_j m_j$ points, and hence the Γ-orbit of A_j consists of $|\Gamma|/d_j m_j$ islands A_j^i. Since for any $j = 1, \ldots, p$, the family of islands A_j^i is symmetric in S, we have:

$$\mathcal{Y} = \sum_j \frac{1}{d_j m_j} |\Gamma|\, \mathcal{W}(S, A_j) \geq \frac{1}{d} |\Gamma| \sum_j \mathcal{W}(U, K_j) = \frac{1}{d} |\Gamma|\, Y, \tag{7.3}$$

where the middle inequality follows from Lemma 10.4.

Let us estimate $\mathcal{Z}$. Let $\mathcal{Z}_j^i$ stand for the extremal width between the island A_j^i and the rest of the boundary of $S \smallsetminus \cup A_k^l$. These widths are realized by the harmonic foliations $\mathcal{F}_j^i$ connecting A_j^i to the rest of the boundary (see §10.3 of Appendix 10). Hence the $\mathcal{Z}_j^i$ are given by the l_1-norms of the associated valid WAD's W_j^i. Now Properties A an B of the canonical WAD's imply:

$$\mathcal{Z} = \sum \mathcal{Z}_j^i = \sum \|W_j^i\|_1 \leq \|W_{\mathrm{can}}^{\mathrm{v+h}}(S \smallsetminus \bigcup_{k,l} A_k^l)\|_1 + 6|\Gamma|^2$$

$$= |\Gamma|\, \|W_{\mathrm{can}}^{\mathrm{v+h}}(U' \smallsetminus \cup K_j')\|_1 + 6|\Gamma|^2. \tag{7.4}$$

(Here $3|\Gamma|^2$ is a rough estimate on the number of arcs in $\cup \operatorname{supp} W_j^i$).

Putting (7.3) and (7.4) together with the Collar Assumption, we obtain the separation property for the family of archipelagos A_j^i (assuming that $Y > 6D!/\xi$):

$$\mathcal{Z} \leq \xi |\Gamma|\, Y + 6|\Gamma|^2 \leq 2\xi |\Gamma| Y \leq 2\xi d \mathcal{Y}.$$

We are now in the position to apply the Quasi-Additivity Law. Together with (7.2) and (7.3), it implies the desired estimate:

$$\frac{1}{d} |\Gamma| X \leq \frac{1}{d} |\Gamma| Y \leq \mathcal{Y} \leq 2C\xi d\, \mathcal{X} = 2C\xi d\, |\Gamma| X'.$$

□

7.3. **Vertical foliation has a definite weight.** We can now prove the main technical result asserting that the vertical foliation constitutes a definite proportion of the canonical foliation.

Lemma 7.1. *Let $f\colon \mathbf{U} \to \mathbf{U}$ be primitively renormalizable ψ-ql map with period p, and let $\mathcal{K}$ be its cycle of little Julia sets. Then there exists $M = M(p)$ such that: If $\|W_{\mathrm{can}}^{\mathrm{v+h}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1 > M$ then*

$$\|W_{\mathrm{can}}^{\mathrm{v}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1 \geq C^{-1} \|W_{\mathrm{can}}^{\mathrm{h}}(\mathbf{U} \smallsetminus \mathcal{K})\|_1,$$

with an absolute constant $C > 0$.

Proof. If $\|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\|_1 \geq \|W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\|_1$, there is nothing to prove. So, we assume that the opposite inequality holds, and then

$$\|W^{\mathrm{h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\|_1 > \frac{1}{2}\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\|_1.$$

Then taking $M = M(p)$ as in Corollary 6.6, we conclude that

$$\|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U}^{10p} \setminus \mathcal{K})\|_1 \geq \frac{1}{2}\|W^{\mathrm{v+h}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\|_1, \tag{7.5}$$

provided $\|W_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\|_1 > 2M$.

We would like to apply the Covering Lemma to the branched covering $f^{10p}\colon (\mathbf{U}^{10p}, \mathcal{K}) \to (\mathbf{U}, \mathcal{K})$. Note that $D \equiv \deg(f^{10p}\colon \mathbf{U}^{10p} \to \mathbf{U}) = 2^{10p}$ depends on p, while $d \equiv \deg(f^{10p}\colon \mathcal{K} \to \mathcal{K}) = 2^{10}$ is absolute. By property A of the canonical WAD's, the moduli of the Covering Lemma get the following meaning:

$$X = W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U}^{10p} \setminus \mathcal{K}) + C(p), \quad X' = W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K}) + C(p),$$

where $C(p)$ stand for different constants depending only on p.

If $W_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K}) > M(p)$ then estimate (7.5) gives us:

$$Y \geq X \geq \frac{1}{2}\|W^{\mathrm{v+h}}_{\mathrm{can}}\|_1(\mathbf{U} \setminus \mathcal{K}) - C(p),$$

which provides us with both the Separation Assumption (with, say, $\xi = 4$) and the assumption $Y > L(p)$, where $L(p)$ is prescribed by the Covering Lemma. Applying the Covering Lemma, we obtain

$$\|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U} \setminus \mathcal{K})\|_1 > (Cd)^{-2}\|W^{\mathrm{v}}_{\mathrm{can}}(\mathbf{U}^{10p} \setminus \mathcal{K})\|_1$$

with an absolute constant C. Together with (7.5), this implies the desired estimate. □

8. From the canonical WAD's to hyperbolic geometry

In this section we will describe how to measure the geometry of a surface using transverse geodesic arcs. We will then show how to compute the lengths of peripheral closed geodesics of the surface using these measurements. Finally, we will show that our new measurements are just the canonical WAD in disguise.

Suppose T is a compact hyperbolic surface with geodesic boundary. The following lemma appears as the Corollary to section 3.3 of [Ab]:

Lemma 8.1. *There is an $\epsilon_0 > 0$ such that any two distinct closed geodesics on T of length at most ϵ_0 are simple and disjoint.*

Let $\boldsymbol{S}$ be a compact hyperbolic surface with geodesic boundary. Then a *transverse geodesic arc* for $\boldsymbol{S}$ is a proper path of minimal length in its proper homotopy class. If $\boldsymbol{\alpha}$ is a path on $\boldsymbol{S}$, it is a transverse geodesic arc if and only if it is a geodesic arc that meets $\partial\boldsymbol{S}$ orthogonally, or, equivalently, the double of $\boldsymbol{\alpha} \cup \overline{\boldsymbol{\alpha}}$ in $\boldsymbol{S} \cup \overline{\boldsymbol{S}}$ is a closed geodesic.

Let S be a compact Riemann surface with boundary, and endow $\mathrm{Int}\, S$ with its Poincaré metric. The peripheral geodesics on $\mathrm{Int}\, S$ bound a compact surface $\boldsymbol{S}$ with geodesic boundary, called the *convex core* of S. There is a homeomorphism $h\colon \boldsymbol{S} \to S$ that is isotopic through embeddings to the inclusion $\boldsymbol{S} \subset S$. We can then form a weighted arc diagram M^S on S as follows: for $\alpha \in \mathcal{A}(S)$, we find the transverse geodesic arc $\boldsymbol{\alpha}$ for $\boldsymbol{S}$ such that $h(\boldsymbol{\alpha}) \sim \alpha$. Then we let $M^S(\alpha) = -\log L(\boldsymbol{\alpha})$ if $L(\boldsymbol{\alpha}) < \epsilon_0/2$, and $M^S(\alpha) = 0$ otherwise.

Then M^S is supported on a set of disjoint arcs, so M^S is a weighted arc-diagram for S. We let $\boldsymbol{M}^S = \bigcup\{\boldsymbol{\alpha} : L(\boldsymbol{\alpha}) < \epsilon_0/2\}$, so $\boldsymbol{M}^S \subset \boldsymbol{S}$ is the union of the short transverse geodesic arcs of $\boldsymbol{S}$.

We will call a non-peripheral simple closed geodesic a *dividing* geodesic. The following result appears in [Ab]:

Lemma 8.2. *Let T be a compact hyperbolic Riemann surface with bounded-length geodesic boundary. Then either T is a pair of pants, or there is a bounded-length dividing geodesic on T.*

We say that a hyperbolic surface T is *symmetric* if it admits an isometric orientation-reversing involution, which we will denote by complex conjugation: "$z \mapsto \overline{z}$". Then we let $E_T = \{z \in T : z = \overline{z}\}$, and E_T will be a union of (simple) closed geodesics and transverse geodesic arcs. (E_T depends implicitly on on the choice of involution. Whenever we say "symmetric hyperbolic surface" we will mean "symmetric hyperbolic surface and choice of involution.") Note that $T \smallsetminus E_T$ has two components, call them A_T and $\overline{A_T}$, which are mapped to each other by $z \mapsto \overline{z}$. We prove a symmetric version of Lemma 8.2:

Lemma 8.3. *Every symmetric compact hyperbolic surface T with bounded-length geodesic boundary has a bounded-length symmetric pair of pants decomposition.*

Proof. It suffices to find a single bounded-length symmetric dividing geodesic on the surface, or a symmetric pair of disjoint bounded-length dividing geodesics, because then we can cut the surface T along that geodesic or pair of geodesics, and repeat.

By Theorem 8.2, unless T is a pair of pants, there is a dividing geodesic γ of bounded length on T. If $\gamma \cap E_T = \emptyset$, then $\gamma \cap \overline{\gamma} = \emptyset$, and we are done. Likewise, if $\gamma \subset E_T$, then γ is symmetric, and we are done. Otherwise, let η be a component of $\gamma \cap \mathrm{Cl}\, A_T$. Then $\eta \cup \overline{\eta}$ is a non-trivial non-peripheral simple closed curve so we let τ be the dividing geodesic homotopic to η. Then $L(\tau) \le 2L(\eta) < 2L(\gamma)$, so τ is the desired object. □

We now prove two basic estimates on transverse geodesic arcs on pairs of pants. We denote by $[x, y, z, r, s, t]$ the right-angled hyperbolic hexagon with lengths x, y, z, r, s, t in that order. We will omit lengths that are not specified, so for example $[a, , b, , c,]$ denotes the right-angled hexagon with alternating side lengths a, b, and c. We first estimate the length of one side in a hyperbolic right-angled hexagon, in terms of the lengths of three alternating sides:

Lemma 8.4. *Let $[a, c', b, , c, ,]$ be a hyperbolic right-angled hexagon, and suppose that $a, b, c \le r$. Then $c' = -\log a - \log b + O(1; r)$.*[10]

Proof. We use the formula (from [F]):

$$\cosh c' = \frac{\cosh c + \cosh a \cosh b}{\sinh a \sinh b} = e^{O(1;r)} \frac{1}{ab} \tag{8.1}$$

and recall that $\cosh^{-1} x = \log x + O(1; t)$ whenever $x \ge t > 0$. □

We let $\mathcal{P}(a, b, c)$ denote the pair of pants with cuff lengths a, b, and c.

Lemma 8.5. We have the following two estimates:

[10]Notation $O(1; r)$ stands for a quantity bounded in terms of r.

(1) If $P = \mathcal{P}(a, b, c)$ is a pair of pants, and γ is the transverse geodesic arc that connects a and b, then

$$|\gamma| = -\log a - \log b + O(1),$$

for $a, b, c \leq C_0$.

(2) If γ is the transverse geodesic arc that connects a and a in $P(a, b, b)$, then

$$|\gamma| = -2\log a + O(1)$$

for $a, b \leq C_0$.

Proof. We prove each of the above:

(1) We cut P along the three pairwise transversals into two right-angled hexagons. By formula (8.1) these hexagons are equal, and hence each has type $[a/2, \gamma, b/2, \, , c/2, \,]$. Apply now Lemma 8.4.

(2) We let η be the transversal from the length a cuff to one other. Then we cut along η and γ to obtain the right-angled hexagon $[a/4, \gamma, a/4, \eta, b, \eta]$, and then apply Lemma 8.4.

□

Given a closed geodesic γ and an arc $\alpha \in \mathcal{A}$, let $\langle \gamma, \alpha \rangle$ stand for the intersection number of γ with α, i.e., the minimal number of intersections of γ with the paths representing α. Given a weighted arc diagram $W = \sum W(\alpha)\alpha$, we can define the intersection number $\langle \gamma, W \rangle = \sum W(\alpha) \langle \gamma, \alpha \rangle$ by linearity.

Theorem 8.6. *Let S be a compact Riemann surface with boundary, and endow* Int S *with its Poincaré metric. Suppose that γ is a peripheral closed geodesic for S. Then*

$$L(\gamma) = 2\left\langle M^S, \gamma \right\rangle + O(1; \chi(S)).$$

Proof. We let $\boldsymbol{S}$ be the convex core of S. We find a symmetric bounded-length pair of pants decomposition for $\boldsymbol{S} \cup \overline{\boldsymbol{S}}$ extending $\boldsymbol{M}^S \cup \bar{\boldsymbol{M}}^S$. Then we can write $\gamma = \bigcup t_i$, where the segments t_i are interior-disjoint, and each is a transverse arc of one of the pairs of pants. Then $L(t_i) = -\log a_i - \log b_i + O(1)$, where a_i and b_i are the lengths of the cuffs that γ connects (possibly the same cuff). Therefore

$$L(\gamma) = 2\sum -\log a_i + O(1)$$

where the a_i are the lengths of the cuffs that γ crosses, counted with multiplicity. But

$$2\sum -\log a_i = 2\left\langle M^S, \gamma \right\rangle + O(1). \tag{8.2}$$

□

We can now relate M^S to W^S_{can} via the following:

Lemma 8.7. *Let Q be a quadrilateral with horizontal sides I_1 and I_2, endowed with the hyperbolic metric. Let Γ_1 and Γ_2 be the hyperbolic geodesics in Π that share the endpoints with I_1 and I_2 respectively. Then*

$$\mathcal{W}(Q) = -\frac{2}{\pi}\log \operatorname{dist}(\Gamma_1, \Gamma_2) + O(1).$$

Proof. We can map Q conformally to the infinite strip

$$\Pi \equiv \{z : 0 < \Im z < \pi\}$$

such that Γ_1 and Γ_2 map to vertical transverse segments with real part 0 and d respectively, where $d \equiv \operatorname{dist}(\Gamma_1, \Gamma_2)$. Then the vertical sides of Q map to $J_1 \equiv [0, d] \times \{0\}$ and $J_2 \equiv [0, d] \times \{\pi\}$. Then $\mathcal{W}(Q)$ is equal to $\mathcal{L}(\Theta)$, where Θ is the family of paths in Π connecting J_1 and J_2. By two applications of the reflection principle (see [A2]), we find that $\mathcal{L}(\Theta) = 4\mathcal{L}(\Theta')$, where Θ' is the family of paths connecting J_1 to the boundary of $\Pi' \equiv \{z : |\Im z| < \pi/2\}$. By the round annulus theorem[McM2]

$$\mathcal{L}(\Theta') = \frac{1}{2\pi} \log \frac{\pi/2}{d} + O(1) = -\frac{1}{2\pi} \log d + O(1)$$

when d is bounded above. The theorem follows. □

We can make the following corollary:

Lemma 8.8. *For any arc $\alpha \in \mathcal{A}(S)$,*

$$|W^S_{can}(\alpha) - \frac{2}{\pi} M^S(\alpha)| < C_0.$$

Proof. Given α, let us consider a lift $\tilde{\alpha}$ to the universal cover. It connects two arcs I_1, I_2 on the circle that cover the boundary curves of S that the endpoints of α lie on. We can lift $h \colon \boldsymbol{S} \to S$ to $\tilde{h} \colon \boldsymbol{R} \to \mathbb{D}$, where $\boldsymbol{R}$ is the convex hull of the limit set of the deck transformation group. Then letting Γ_1, Γ_2 connect the endpoints of I_1, I_2 respectively, we find that the transverse geodesic arc $\boldsymbol{\alpha}$ lifts to the common perpendicular segment to Γ_1 and Γ_2. So $\operatorname{dist}(\Gamma_1, \Gamma_2) = L(\boldsymbol{\alpha})$, and the Lemma follows from Lemma 8.7. □

Then the main result for this section follows immediately from the above and Theorem 8.6:

Corollary 8.9. Let S be a compact Riemann surface with boundary, and endow $\operatorname{Int} S$ with its Poincaré metric. Let γ be a peripheral closed geodesic in $\operatorname{Int} S$. Then

$$L(\gamma) = \pi \langle W_{can}(S), \gamma \rangle + O(1; \chi(S)).$$

9. Improving of the moduli

We are now ready to show that the modulus of a ψ-quadratic-like map is improving under the renormalization.

Let $f \colon \mathbf{U} \to \mathbf{U}$ be a ψ-quadratic-like map with filled Julia set K. The simple closed geodesic γ in the annulus $\mathbf{U} \smallsetminus K$ is called the *geodesic associated with* f. Let $|\gamma|$ stand for its hyperbolic length in $\mathbf{U} \smallsetminus K$.

Theorem 9.1. *For any $\lambda > 1$, there exists $\underline{p}$ such that for any $\bar{p} \geq \underline{p}$, there exists $L(\bar{p})$ with the following property. Let $f \colon \mathbf{U} \to \mathbf{U}$ be primitively renormalizable ψ-quadratic-like map with period p such that $\underline{p} \leq p \leq \bar{p}$. Let γ and γ' be the geodesics associated with f and $f' = Rf$ respectively. Then:*

$$|\gamma'| > L(\bar{p}) \Rightarrow |\gamma| > \lambda |\gamma'|.$$

Proof. Let K' be the filled Julia set of f', $K'_j = f^j(K')$, $j = 0, 1 \ldots, p-1$, and $\mathcal{K}' = \cup K'_j$. Let γ'_j be the peripheral simple closed geodesics in $\mathbf{U} \setminus \mathcal{K}'$ going around K'_j, and let Γ be peripheral closed geodesic in $\mathbf{U} \setminus \mathcal{K}'$ homotopic to $\partial \mathbf{U}$. We let $W_{\text{can}} \equiv W_{\text{can}}(\mathbf{U} \setminus \mathcal{K}')$.

The geodesic Γ intersects each vertical arc once and does not intersect horizontal arcs. By Corollary 8.9,

$$|\Gamma| \geq c \|W^{\text{v}}_{\text{can}}\|_1 + O(1). \tag{9.1}$$

Let $W_{\text{can}}|\, j$ be the part of W_{can} supported on the arcs landing at K'_j. The geodesic γ'_j does not intersect $W_{\text{can}} - W_{\text{can}}|\, j$ and intersects each arc of $\operatorname{supp} W_{\text{can}}|\, j$ once. By Corollary 8.9,

$$|\gamma'_j| = c\|W_{\text{can}}|\, j\|_1 + O(1). \tag{9.2}$$

But by the Schwarz Lemma, the geodesics γ'_j have comparable lengths (see [McM1, Theorem 9.3]):

$$\frac{1}{2}|\gamma'_0| \leq |\gamma'_j| \leq |\gamma'|.$$

Putting this together with (9.2), we see that all the $\|W_{\text{can}}|\, j\|_1$ are also comparable, provided $|\gamma'_0|$ is sufficiently big (bigger than some absolute L_0). Hence

$$\|W_{\text{can}}(0)\|_1 \asymp \frac{1}{p}\|W^{\text{v+h}}_{\text{can}}\|_1,$$

and together with (9.2), we obtain:

$$|\gamma'_0| \asymp \frac{1}{p}\|W^{\text{v+h}}_{\text{can}}\|_1.$$

But by Lemma 7.1, $\|W_{\text{can}}\|_1$ is comparable with $\|W^{\text{v}}_{\text{can}}\|_1$, so that,

$$|\gamma'_0| \leq \frac{C}{p}\|W^{\text{v}}_{\text{can}}\|_1,$$

provided $\|W^{\text{v+h}}_{\text{can}}\|_1 > M(\bar{p})$. Together with (9.1), this implies that

$$|\gamma'_0| \leq \frac{1}{\lambda}|\Gamma|,$$

provided $\underline{p} > 2C/c$, $\|W^{\text{v+h}}_{\text{can}}\|_1 > M(\bar{p})$, and $|\gamma'_0|$ is bigger than some absolute L_0. But in view of (9.2), the last two conditions are satisfied if $|\gamma'_0| \geq L(\bar{p})$.

What is left is to notice is that $|\gamma'_0| = |\gamma_0|$, while by the Schwarz Lemma, $|\Gamma| \leq |\gamma|$ (since $K \supset \mathcal{K}'$). $\square$

Corollary 9.2. *Let f be an infinitely renormalizable ψ-quadratic-like map of r-bounded primitive type. Let γ_n be the hyperbolic geodesics associated with the canonical renormalizations of f. Then*

$$|\gamma_n| \leq K(r).$$

Proof. Below $\underline{p}$ and $L(\bar{p})$ come from Lemma 9.1. Let f_n be the canonical ψ-ql renormalizations of f.

Take a natural number k such that $2^k \geq \underline{p}$, and let $\bar{p} = r^k$. Let us show that $\limsup |\gamma_n| \leq L = L(\bar{p})$. Indeed, otherwise, there would be a sequence $n(i) \to \infty$ such that $|\gamma_{n(i)}| > L$.

The map $f_{n(i)}$ is the primitive p_i-renormalization of $f_{n(i-k)}$ with period $p_i \in [\underline{p}, \bar{p}]$. By Theorem 9.1,

$$|\gamma_{n(i-k)}| \geq 2|\gamma_{n(i)}|.$$

Iterating this estimate backwards, we conclude that $|\gamma_m| \geq 2^{i/k-1}L$ for some level $m = m(i) < k$, which is certainly impossible. $\square$

Corollary 9.3. *Let f be an infinitely renormalizable ψ-quadratic-like map of r-bounded primitive type. Then there is a sequence of quadratic-like renormalizations $f_n: U_n \to V_n$ with*

$$\liminf \operatorname{mod}(V_n \smallsetminus U_n) \geq \varepsilon(r) > 0.$$

Proof. Let $f_n: (\mathbf{U}_n, K_n) \to (\mathbf{U}_n, K_n)$ be, as above, the canonical ψ-ql renormalizations of f with associated geodesics γ_n. Then

$$\operatorname{mod}(\mathbf{U}_n \smallsetminus K_n) = \frac{\pi}{|\gamma_n|},$$

so that, by the previous corollary, $\liminf \operatorname{mod}(\mathbf{U}_n \smallsetminus K_n) \geq \pi/K(r)$. By Lemma 2.4, the maps f_n admit quadratic-like restrictions $U_n \to V_n$ with the desired property $\square$

And this completes the proof of the Main Theorem.

10. Appendix A: Extremal length and width

There is a worth of sources containing background material on extremal length, see, e.g., the book of Ahlfors [A1]. We will briefly summarize the necessary minimum (see also the Appendix of [KL1]).

10.1. **Definitions.** Let $\mathcal{G}$ be a family of curves on a Riemann surface U. Given a (measurable) conformal metric $\mu = \mu(z)|dz|$ on U, let

$$\mu(\mathcal{G}) = \inf_{\gamma \in \mathcal{G}} \mu(\gamma).$$

where $\mu(\gamma)$ stands for the μ-length of γ. The length of $\mathcal{G}$ with respect to μ is defined as

$$\mathcal{L}_\mu(\mathcal{G}) = \frac{\mu(\mathcal{G})^2}{\operatorname{area}_\mu(U)},$$

where area_μ is an area form of μ. Taking the supremum over all conformal metrics μ, we obtain the *extremal length* $\mathcal{L}(\mathcal{G})$ of the family $\mathcal{G}$.

The *extremal width* is the inverse of the extremal length:

$$\mathcal{W}(\mathcal{G}) = \mathcal{L}^{-1}(\mathcal{G}).$$

It can be also defined as follows. Consider all conformal metrics μ such that $\mu(\gamma) \geq 1$ for any $\gamma \in \mathcal{G}$. Then $\mathcal{W}(\mathcal{G})$ is the infimum of the areas $\operatorname{area}_\mu(U)$ of all such metrics.

10.2. **Electric circuits laws.** We say that a family $\mathcal{G}$ of curves *overflows* a family $\mathcal{G}'$ if any curve of $\mathcal{G}$ contains some curve of $\mathcal{G}$. Let us say that $\mathcal{G}$ *disjointly overflows* two families, $\mathcal{G}_1$ and $\mathcal{G}_2$, if any curve of $\gamma \in \mathcal{G}$ contains the disjoint union $\gamma_1 \sqcup \gamma_2$ of two curves $\gamma_i \in \mathcal{H}_i$.

The following crucial properties of the extremal length and width show that the former behaves like the resistance in electric circuits, while the latter behaves like conductance.

Series Law. *Let $\mathcal{G}$ be a family of curves that disjointly overflows two other families, $\mathcal{G}_1$ and $\mathcal{G}_2$. Then*

$$\mathcal{L}(\mathcal{G}) \geq \mathcal{L}(\mathcal{G}_1) + \mathcal{L}(\mathcal{G}_2),$$

or equivalently,

$$\mathcal{W}(\mathcal{G}) \leq \mathcal{W}(\mathcal{G}_1) \bigoplus \mathcal{W}(\mathcal{G}_2).$$

Parallel Law. *For any two families $\mathcal{G}_1$ and $\mathcal{G}_2$ of curves we have:*

$$\mathcal{W}(\mathcal{G}_1 \cup \mathcal{G}_2) \leq \mathcal{W}(\mathcal{G}_1) + \mathcal{W}(\mathcal{G}_2).$$

If $\mathcal{G}_1$ and $\mathcal{G}_2$ are contained in two disjoint open sets, then

$$\mathcal{W}(\mathcal{G}_1 \cup \mathcal{G}_2) = \mathcal{W}(\mathcal{G}_1) + \mathcal{W}(\mathcal{G}_2)$$

10.3. **Transformation rules.** Both extremal length and extremal width are conformal invariants. More generally, we have:

Lemma 10.1. *Let $f: U \to V$ be a holomorphic map between two Riemann surfaces, and let $\mathcal{G}$ be a family of curves on U. Then*

$$\mathcal{L}(f(\mathcal{G})) \geq \mathcal{L}(\mathcal{G}).$$

See Lemma 4.1 of [KL1] for a proof.

Corollary 10.2. *Under the circumstances of the previous lemma, let $\mathcal{H}$ be a family of curves in V satisfying the following lifting property: any curve $\gamma \in \mathcal{H}$ contains an arc that lifts to some curve in $\mathcal{G}$. Then $\mathcal{L}(\mathcal{H}) \geq \mathcal{L}(\mathcal{G})$.*

Proof. The lifting property means that the family $\mathcal{H}$ overflows the family $f(\mathcal{G})$. Hence $\mathcal{L}(\mathcal{H}) \geq \mathcal{L}(f(\mathcal{G}))$, and the conclusion follows. □

Corollary 10.3. *Let Q and Q' be two quadrilaterals, and let $e: Q \to Q'$ be a holomorphic map that maps the horizontal sides of Q to the horizontal sides of Q'. Then $\mathcal{W}(Q) \leq \mathcal{W}(Q')$.*

Proof. Let $\mathcal{G}$ (resp. $\mathcal{G}'$) be the family of horizontal curves in Q (resp., in Q'). Since the horizontal sides of Q are mapped to the horizontal sides of Q', these families satisfy the lifting property of the previous Corollary. Hence $\mathcal{L}(\mathcal{G}) \leq \mathcal{L}(\mathcal{G}')$, and we are done. □

Lemma 10.4. *Let $f: U \to V$ be a branched covering between two compact Riemann surfaces with boundary. Let A be an archipelago in U, $B = f(A)$, and assume that $f: A \to B$ is a branched covering of degree d. Then*

$$\operatorname{mod}(V, B) \geq d \operatorname{mod}(U, A).$$

See Lemma 4.3 of [KL1] for a proof.

Given two compact subsets A and B in a Riemann surface S, let $\mathcal{W}_S(A, B)$ stand for the *extremal width* between them, i.e., the extremal width of the family of curves connecting A to B.

Lemma 10.5. *Let S and S' be two compact Riemann surfaces with boundary. and let $f: S \to S'$ be a branched covering of degree D. Let $S' = A' \sqcup B'$, where A' and B' are closed, and let $A = f^{-1}(A')$, $B = f^{-1}(B')$. Then*

$$\mathcal{W}_S(A, B) = D\, \mathcal{W}(A', B').$$

See [A1] for a proof. It makes use of the fact that the extremal width $\mathcal{W}(A, B)$ is achieved on the *harmonic foliation* $\mathcal{F} = \mathcal{F}_S(A, B)$ connecting A and B, i.e., the gradient foliation of the harmonic function ω vanishing on 0 and equal to 1 on B. Hence $\mathcal{W}_S(A, B)$ is equal to the l_1-norm of the associated WAD $W_{\mathcal{F}}$.

10.4. **Non-Intersection Principle.** The following important principle says that two wide quadrilaterals cannot go non-trivially one across the other:

Lemma 10.6. *Let us consider two quadrilaterals, Q_1 and Q_2, endowed with the vertical foliations $\mathcal{F}_1$ and $\mathcal{F}_2$. If $\mathcal{W}(Q_i) \geq 1$ then*
• *either there exists a pair of disjoint leaves γ_i of the foliations $\mathcal{F}_i$;*
• *or $\mathcal{W}(Q_1) = \mathcal{W}(Q_2) = 1$, and the rectangles perfectly match in the sense that the vertical sides of one of them coincide with the horizontal sides of the other.*

Proof. Assume that the first option is violated, so that, every leaf of $\mathcal{F}_1$ crosses every leaf of $\mathcal{F}_2$. Let us uniformize our rectangles by standard rectangles, $\phi_i \colon \mathbf{Q}(a_i) \to Q_i$, where $a_i \geq 1$. Let λ_i be the Euclidean metrics on the $\mathbf{Q}(a_i)$, and let $\mu_i = (\phi_i)_*(\lambda_i)$. Since every vertical leave γ of Q_2 crosses every vertical leaf of Q_1, $\mu_1(\gamma) \geq a_1$. Hence $a_2 = \mathcal{W}(Q_2) \leq 1/a_1$, which implies that $a_1 = a_2 = 1$. Thus, both $\mathbf{Q}(a_i) \equiv \mathbf{Q}$ are the squares.

Moreover, μ_1 must be the extremal metric for $\mathcal{F}_2$. Since the extremal metric is unique (up to scaling), we conclude that $\mu_1 = \mu_2$. Hence $\phi_2^{-1} \circ \phi_1 \colon \mathbf{Q} \to \mathbf{Q}$ is the isometry of the square, and the conclusion follows. □

11. Appendix B: Elements of electric engineering

11.1. **Potentials, currents and conductances.** *Electric circuit* $\mathcal{C}$ is

- A connected graph Γ with two marked vertices (*battery*). We let $\mathcal{E} = \mathcal{E}(\Gamma)$ be the set of edges of Γ, $\mathcal{V} = \mathcal{V}(\Gamma)$ be the set of its vertices, and $\mathcal{B} = \{a, b\} \subset \mathcal{V}$ be the battery.
- A *conductance* vector $W = \sum_{e \in \mathcal{E}} W(e)\, e$, where $W(e) > 0$ for all $e \in \mathcal{E}$.

The edges of Γ are interpreted as *resistors* of the circuit with conductances $W(e)$. The inverse numbers $R(e) = W(e)^{-1}$ as their *resistances*.

We write $x \sim y$ for two neighboring vertices of Γ. The vertices of $\mathcal{V} \smallsetminus \mathcal{B}$ will also be called *internal*. If we forget the battery $\mathcal{B}$, we call the circuit "unplugged".

A *potential distribution* on $\mathcal{C}$ is a function $U \colon \mathcal{V} \to \mathbb{R}$.

Let $\mathcal{E}^*$ stand for the set of all possible *oriented* edges $\mathcal{E}$. An oriented edge e^* can be written as $[x, y]$ where x, y are its endpoints ordered according to the orientation of e. We also write $-e^*$ for the edge e^* with the opposite orientation.

A *current* on $\mathcal{C}$ is an odd function $I \colon \mathcal{E}^* \to \mathbb{R}$, i.e., $I(-e^*) = -I(e^*)$.

A potential U induces potential differences

$$dU[x, y] = U(y) - U(x)$$

on the oriented edges of Γ (negative *coboundary* of U). It forces current

$$I(e^*) = -W(e)\, dUe^*, \quad e^* \in \mathcal{E}^*,$$

to run through the resistors. The energy $E(e)$ of this current is equal to $I(e)\, dU(e) = W(e)\, (dU(e))^2$ (note that it is independent of the orientation of e), so that, the total energy of this potential distribution is equal to

$$E = E(U) = \sum_{e \in \mathcal{E}} W(e)\, (dU(e))^2.$$

The quantity $\mathbf{U} = U(a) - U(b)$ is called the *battery potential*.

11.2. **Equilibrium.** Let us define the *boundary* of the current I as the following function on the vertices:

$$\partial I(x) = \sum_{y \sim x} I[x, y].$$

We say that the potential distribution is in *equilibrium*, U_{eq}, if the current is *conserved*, i.e., $\partial I \equiv 0$ on $\mathcal{V} \setminus \mathcal{B}$. (In other words, a conserved current is a relative 1-cycle on $(\Gamma, \mathcal{B})$). It is equivalent to saying that the potential U is "W-harmonic" on $\Gamma \setminus \mathcal{B}$.

We say that a potential distribution U is *normalized* if $U(a) = 1$, $U(b) = 0$.

Lemma 11.1. *There exists a unique normalized equilibrium potential distribution on* $\mathcal{C}$*. This distribution is energy minimizing.*

Proof. The restriction of the energy function E to the affine subspace

$$\mathcal{A} = \{U \in \mathbb{R}^{|\mathcal{E}|} : U(a) = 1,\ U(b) = 0\}$$

is a positive quadratic function. One can easily check that $E(U) \to +\infty$ as $U \to \infty$, $U \in \mathcal{A}$, so that, U has a global minimum U_{eq} on $\mathcal{A}$. Moreover, E is strictly convex, an hence can have at most one critical point. Hence U_{eq} is the only critical point of U on $\mathcal{A}$. Finally the criticality condition gives exactly the conservation law for the corresponding current. ☐

Remark 1. Since the energy function $E(U)$ depends only on the potential differences, it is invariant under translations $U \mapsto U + t\mathbf{1}$, $t \in \mathbb{R}$, where $\mathbf{1} \equiv 1$. Thus, we can always normalize U so that $U(b) = 0$. Since $E(U)$ is homogeneous of order 2 in U, normalization $\mathbf{U} = \lambda$ would replace the equilibrium distribution U_{eq} by $U^{\lambda}_{eq} = \lambda U_{eq}$. Then the equilibrium current would scale proportionally: $I^{\lambda}_{eq} = \lambda I_{eq}$.

Remark 2. The above lemma is just a solution of the Dirichlet boundary problem by minimizing the Dirichlet integral.

Lemma 11.2. *For the equilibrium current* I_{eq}*, we have* $\partial I_{eq}(a) = -\partial I_{eq}(b)$.

Proof. We have:

$$0 = \sum_{e^* \in \mathcal{E}^*} I_{eq}(e^*) = \sum_{x \in \mathcal{V}} \partial I_{eq}(x) = \partial I_{eq}(a) + \partial I_{eq}(b),$$

where the first equality is valid since I is odd, the second is a rearrangement of terms, and the last one comes from the conservation law. ☐

The *total equilibrium current* of the circuit $\mathcal{C}$ with battery potential $\mathbf{U} = \lambda$ is defined as $\mathbf{I} = I^{\lambda}_{eq}(a) = -I^{\lambda}_{eq}(b)$. It depends linearly on the battery potential, so we can define the *total conductance* of $\mathcal{C}$ as

$$\mathbf{W} = \mathbf{W}(\mathcal{C}) = \frac{\mathbf{I}}{\mathbf{U}}.$$

The *total Resistance* of the circuit is the inverse quantity: $\mathbf{R} = 1/\mathbf{W}$.

Let $\mathbf{E} = E(U^{\lambda}_{eq})$ stand for the equilibrium energy of the circuit.

Lemma 11.3. *At the normalized equilibrium, we have:* $\mathbf{E} = \mathbf{W}$.

Proof. Let $\mathcal{A}_0 = \{U \in \mathbb{R}^{|\mathcal{E}|} : U(b) = 0\} = \mathbb{R}^{|\mathcal{E}|-1}$. Then the equilibrium state is the critical point of the quadratic form $E(U) = (QU, U)$ on $\mathcal{A}_0$ subject of the restriction $U(a) = 1$ (here Q is the matrix of $E|\mathcal{A}_0$). By the Lagrange multipliers method, this stationary point satisfies the equation

$$\frac{\partial E}{\partial U} \equiv 2QU = \lambda V,$$

where V is a basic vector in $R^{|\mathcal{E}|-1}$ with $V(a) = 1$, $V(x) = 0$ for $x \in \mathcal{V} \smallsetminus \mathcal{B}$. From here we conclude:

(i) $2E(U) = 2(QU, U) = \lambda(V, U) = \lambda U(a) = \lambda$;

(ii) $\dfrac{\partial E}{\partial U(a)} = \lambda V(a) = \lambda$.

But

$$\frac{\partial E}{\partial U(a)} = \frac{\partial}{\partial U(a)} \sum_{x \sim a} W[a, x](U(a) - U(x))^2 = 2 \sum_{x \sim a} I[a, x] = 2\mathbf{W},$$

and we are done. □

Remark. So, in the normalized situation (when $\mathbf{U} = 1$), we have

$$\mathbf{E} = \mathbf{I} = \mathbf{W}. \tag{11.1}$$

Since $\mathbf{I}$ is proportional to $\mathbf{U}$, $\mathbf{E}$ quadratically depends on $\mathbf{U}$, and $\mathbf{W}$ is independent of $\mathbf{U}$, we obtain by scaling the following physically obvious formulas:

$$\mathbf{E} = \mathbf{U} \cdot \mathbf{I}, \quad \mathbf{W} = \frac{\mathbf{I}}{\mathbf{U}}.$$

11.3. **Series and Parallel Laws.** Given two circuits $\mathcal{C}_1$ and $\mathcal{C}_2$, we can put them in *series*, that is, to match the terminal battery pole of $\mathcal{C}_1$ to the initial battery pole of $\mathcal{C}_2$, and to declare the "free" battery poles a new battery of this "connected sum". More formally, let $\mathcal{B}_i = \{a_i, b_i\}$ be the battery of $\mathcal{C}_i = (\Gamma_i, W_i)$. Then we let $\mathcal{C} = C_1 \sqcup_{a_2 = b_1} C_2$ with the battery $\{a_1, b_2\}$ and the conductance vector: $W = W_1 + W_2$. (direct sum).

Lemma 11.4 (Series law)**.** *If $\mathcal{C}$ is a series of two circuits $\mathcal{C}_1$ and $\mathcal{C}_2$ with conductances $\mathbf{W}$, $\mathbf{W}_1$ and $\mathbf{W}_2$ respectively, then*

$$\mathbf{W} = \mathbf{W}_1 \oplus \mathbf{W}_2.$$

Proof. Let us consider the equilibrium state (U, I) of the circuit $\mathcal{C}$. Then the conservation law is valid for both sub-circuits $\mathcal{C}_1$ and $\mathcal{C}_2$, so that they are both in the equilibrium state, with the battery potentials $\mathbf{U}_1 = 1 - U(c)$ and $\mathbf{U}_2 = U(c)$, where $c = b_1 = a_2$. Let $\mathbf{I}_i$ be the corresponding equilibrium currents through the $\mathcal{C}_i$. By the current conservation at vertex c, $\mathbf{I}_1 = \mathbf{I}_2$. In fact, $\mathbf{I}_1 = \partial I(a_1) = \mathbf{I}$ and $\mathbf{I}_2 = -\partial I(b_2) = \mathbf{I}$. Hence

$$\frac{1}{\mathbf{W}_1} = \frac{\mathbf{U}_1}{\mathbf{I}}, \quad \frac{1}{\mathbf{W}_2} = \frac{\mathbf{U}_2}{\mathbf{I}}.$$

Summing these up, we obtain the desired. □

Given two circuits as above, we can also put them in *parallel*, that is, to identify the two pairs of poles as follows: $a_1 \sim a_2$, $b_1 \sim b_2$.

Lemma 11.5 (Parallel Law)**.** *If $\mathcal{C}$ is a parallel of two circuits $\mathcal{C}_1$ and $\mathcal{C}_2$ as above, then*

$$\mathbf{W} = \mathbf{W}_1 + \mathbf{W}_2.$$

11.4. **Quotients.** Let us have a finite family of electric circuits $\mathcal{C}_i$ based on graphs Γ_i with batteries $\mathcal{B}_i = \{a_i, b_i\}$ and conductance vectors $\mathcal{W}_i$. Let us identify certain vertices of the disjoint union $\sqcup\Gamma_i$ so that the batteries of different $\mathcal{C}_i$'s get identified ($a_i = a_j$, $b_i = b_j$) and no internal vertices get identified with a battery vertex. If in the quotient graph there are several edges $e_{ij} \in \mathcal{E}_i \equiv \mathcal{E}(\Gamma_i)$ connecting the same pair of vertices, let us identify those edges, too. We obtain a graph Γ. For the edge e of $\mathcal{E} \equiv \mathcal{E}(\Gamma)$ obtained this way, let

$$W(e) = \sum_{e_{ij} \sim e} W(e_{ij}).$$

In this way we obtain a new electric circuit $\mathcal{C} = (\Gamma, \mathcal{B}, \mathcal{W})$ called a quotient of $\sqcup\mathcal{C}_i$.

Let $\mathbf{W} = \mathcal{W}(\mathcal{C})$, $\mathbf{W}_i = \mathcal{W}(\mathcal{C}_i)$.

Lemma 11.6. *If $\mathcal{C}$ is a quotient of $\sqcup\mathcal{C}_i$ then $\mathbf{W} \geq \sum \mathbf{W}_i$.*

Proof. Let U be the normalized equilibrium potential for $\mathcal{C}$.
By Lemma 11.3, its energy $E(U)$ is equal to $\mathbf{W}$.

The potential U lifts to normalized potentials U_i for the circuits $\mathcal{C}_i$'s. By the same lemma, $\mathbf{W}_i \leq E(U_i)$. On the other hand,

$$\sum_i E(U_i) = \sum_i \sum_{e_{ij} \in \mathcal{E}_i} W(e_{ij}) dU_i(e_{ij})^2 = \sum_{e \in \mathcal{E}} dU(e)^2 \sum_{e_{ij} \sim e} W(e_{ij}) = E(U).$$

Putting these pieces together, we obtain the desired inequality. □

Remark 11.1. The above inequality has a clear physical meaning: taking a quotient of a circuit gives more choices for the current to flow, which increases total conductance.

The following arithmetic inequality can be proven by interpreting it in terms of electric circuits:

Lemma 11.7. *Let us consider finite sets of positive numbers w_i and v_{ij}, $1 \leq i, j \leq n$, such that $w_i \leq \oplus_j v_{ij}$. Let $w = \sum w_i$ and $v_j = \sum_i v_{ij}$. Then $w \leq \oplus v_j$.*

Proof. Let us consider resistors V_{ij} with conductances v_{ij}. Let $\mathcal{C}_i$ be an electric circuit obtained by plugging the resistors V_{ij} in series. Then

$$\mathbf{W}(\mathcal{C}_i) = \bigoplus_j v_{ij} \geq w_i.$$

Let us also consider the quotient $\mathcal{C}$ of $\sqcup\mathcal{C}_i$ obtained by identifying the endpoints of the resistors V_{ij} with the same j. In other words, we plug the V_{ij}'s with the same j in parallel obtaining circuits V_j, and then plug the V_i's in series. Then

$$\mathbf{W}(\mathcal{C}) = \bigoplus \mathbf{W}(V_j) = \oplus v_j.$$

By the previous lemma, $\sum \mathbf{W}(\mathcal{C}_i) \leq \mathbf{W}(\mathcal{C})$, which boils down to the desired estimate. □

Remark 11.2. So, the signs of ordinary and harmonic sums can be interchanged following the rule:

$$\sum_i \bigoplus_j v_{ij} \leq \bigoplus_j \sum_i v_{ij}.$$

11.5. **Local conductances.** Given a vertex $x \in \mathcal{V}(\Gamma)$, we let

$$W|x = \sum_{y \sim x} W[x, y],$$

and call it the *local conductance at* x.

Lemma 11.8. *We have:* $\mathbf{W} \le \min(W|a, W|b)$.

Proof. By the Maximum Principle for harmonic functions, $0 \le U_{eq}(x) \le 1$ for any $x \in \Gamma$. Hence $|I_{eq}(e^*)| \le W(e)$ for any $e^* \in \mathcal{E}^*$. Together with (11.1), this implies:

$$\mathbf{W} = \mathbf{I} = \sum_{x \sim a} I_{eq}[x, a] \le W|a,$$

and similarly for $W|b$. $\square$

11.6. **Domination.** Let us consider two unplugged electric circuits $\mathcal{C} = (\Gamma, W)$ and $\mathcal{C}' = (\Gamma', W')$ such that:

- Γ' is obtained from Γ by replacing edges $e \in \mathcal{E}(\Gamma)$ with some graphs $\Gamma'(e)$;
- Letting $\mathcal{C}'(e)$ be the restriction of $\mathcal{C}'$ to $\Gamma'(e)$ with battery ∂e, we have: $\mathbf{W}(\mathcal{C}'(e)) \ge W(e)$ for any $e \in \mathcal{E}(\Gamma)$.

Under these circumstances we say that $\mathcal{C}'$ *dominates* $\mathcal{C}$, $\mathcal{C}' \multimap \mathcal{C}$.

Lemma 11.9. *If* $\mathcal{C}' \multimap \mathcal{C}$ *then* $W'|x \ge W|x$ *for any* $x \in \mathcal{V}(\Gamma)$.

Proof. Indeed, by Lemma 11.8, we have:

$$W'|x = \sum_{\partial e' \ni x} W'(e') \ge \sum_{\partial e \ni x} \mathbf{W}(\mathcal{C}'(e)) \ge \sum_{\partial e \ni x} W(e) = W|x,$$

where the summation is taken over $e \in \mathcal{E}(\Gamma)$, $e' \in \mathcal{E}(\Gamma')$. $\square$

11.7. **Trees of complete graphs.** In this section we will consider a special class of electric circuits based on trees of complete graphs (TCG). A TCG is an object that can be constructed inductively by the following rules:

- Any complete graph is a TCG;
- If Γ, Γ' are TCG's, $v \in \mathcal{V}(\Gamma)$, $v' \in \mathcal{V}(\Gamma')$, then $\Gamma \underset{v=v'}{\sqcup} \Gamma'$ is a tree of complete graphs as well.

A TCG is called an *interval of complete graphs* if any complete graph involved has a common vertex with at most two other complete graphs, and now three complete graphs have a common vertex.

Given three vertices x, y, z in a graph Γ, we say that a vertex y *separates* x *from* z if x and z belong to different components of $\Gamma \smallsetminus \{y\}$.

We say that a sequence of vertices $(x_0, x_1, \dots, x_d)$ form a *chain in* Γ if x_i separates x_{i-1} from x_{i+1} for each $i = 1, \dots, d-1$. Let $SS(x, y)$ stand for the set of vertices separating x from y.

The reader can entertain himself by verifying the following fact:

Lemma 11.10. *Let* Γ *be a TCG. Then for any two vertices* $x, y \in \mathcal{V}(\Gamma)$, *the set* $SS(x, y) \cup \{x, y\}$ *can be uniquely ordered to form a chain* $(x = x_0, x_1, \dots, x_d = y)$. *Moreover, for any* $i = 0, \dots, d-1$, *the vertices* x_i *and* x_{i+1} *belong to the same complete graph* $\Gamma_{k(i)}$, *and these graphs form an interval of complete graphs.*

We call it "the chain connecting x to y", and we let $d_\Gamma(x,y)=d$.

Lemma 11.11. *Let us consider an electric circuit $\mathcal{C}$ based on a tree of complete graphs, and let $(a=x_0,x_1,\dots,x_d=b)$ be the chain connecting the poles of the battery. Then*

$$\mathbf{W}\le\bigoplus_{i=1}^{d}W|x_i\le\frac{1}{d_\Gamma(a,b)}\max_{x\in\Gamma}W|x.$$

Proof. The second inequality is trivial, so we only need to prove the first one.

Let G_i be the component of $\Gamma\smallsetminus\{x_i,x_{i+1}\}$ containing the edge (x_i,x_{i+1}). Let $\mathcal{C}_i$ be the restriction of the electric circuit $\mathcal{C}$ to $G_i\cup\{x_i,x_{i+1}\}$ with battery $\{x_i,x_{i+1}\}$. By the Series Law and Lemma 11.8,

$$\mathbf{W}\le\bigoplus_{i=1}^{d}\mathbf{W}(\mathcal{C}_i)\le\bigoplus_{i=1}^{d}W|x_i.$$

□

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